siding, was addressed by Professor A. G. Webster on "Mathematics of Warfare." As this well known mathematical physicist is on the U. S. Naval Consulting Board, he was well equipped to tell of current developments in the theory of ballistics. He described the elementary principles of ballistics for long range trajectories, summarizing the contributions to the subject of such men as Danforth, Siacci and Chapel. After developing the "ballistic function" and its differential equation, he closed by touching more briefly on the outstanding problems of interior ballistics. The address when published in its full form will furnish a notable chapter in this theory.

A valuable and suggestive program was arranged by the program committee under the chairmanship of Professor Archibald. A still greater interest and concentration of effort centered about the almost continuous conferences over the formulation of war courses in mathematics. Before this report appears, all institutions concerned will have been informed as to the United States government's recommendations or instructions in this respect. It will however remain as a unique feature of this meeting that thirty or forty professors in collegiate mathematics from representative schools in all parts of the country except the far west met in an informal capacity and after much discussion and earnest consideration came to a united view as to the character and contents of three twelve-week terms for the general military student in college, two twelve-week terms in navigation, and an eight-week course in descriptive geometry, besides suggestions as to further second year courses, so far as there will be men to attend these. The courses as formulated were transmitted through Major Huntington as recommendations made by this unofficial conference to the educational committee of the war department and will doubtless have been incorporated in whole or in part into the suggestions issued by that department and already familiar to the readers of the MONTHLY before the appearance of this report.

ORDER OF TOPICS ON THE SEPARATE PROGRAM.

(1) "The Teaching of Curve Tracing." PROFESSOR F. W. OWENS, Cornell University.

(2) "A Formula in Combinatorial Analysis." PROFESSOR J. W. YOUNG, Dartmouth College.

(3) "Trigonometric Functions—of what?" PROFESSOR W. B. CARVER, Cornell University.

(4) "Firing Data at Yale." J. K. WHITTEMORE, Yale University.

(5) "A Combined Course in Mathematics for College Freshmen." PROFESSOR A. S. GALE, University of Rochester.

(6) Report of the Committee on Mathematical Requirements. PROFESSOR J. W. YOUNG, Chairman.

(7) Presidential Retiring Address: "Plans for a History of Mathematics of the Nineteenth Century." PROFESSOR FLORIAN CAJORI, University of California.

(8) "Some Experiments in the Teaching of Descriptive Geometry." PROFESSOR F. L. KENNEDY, Harvard University.

Abstracts, numbered to correspond with the numbers on the foregoing program, are printed below, together with reports of some further informal discussions.

ABSTRACTS OF PAPERS.

(1) Professor Owens in his paper on the "Teaching of Curve Tracing" urged that the light thrown upon the whole notion of functionality by the introduction of the graph may be much intensified by a much greater use of graphical methods in the *construction* of the curve which exhibits the function. He exhibited some simple methods which he has found to be particularly helpful to students in constructing graphs for functions which are or may be given explicitly, and which make especially clear the genesis of the more complicated function from more simple ones by processes indicated in the expression for the function.

(2) Professor Young considers a class $K = (k_1, k_2, \ldots, k_n)$ on N objects. Any combination C_s of s elements of K, together with the combination $\overline{C}_s$ of the remaining $N - s$ objects is called a *partition* of K, C_s and $\overline{C}_s$ being called the *sides* of the partition. A pair, triple, . . . , i-ad of elements of K is said to occur in a partition if the pair, triple, . . . , i-ad occurs in either side of the partition. A *system* S_m of partitions $(C_{s_i}, \overline{C}_{s_i})$ $(i = 1, 2, \ldots, m)$ consist of any set of m partitions; the s_i are not assumed to be all equal, though they may be. The symbol ξ_i is used to represent any i-ad of K; P_{ξ_i} to represent the number of times the i-ad ξ_i occurs in a given S_m. Thus $P_{k_1 k_2}$ means the number of times the pair $k_1\ k_2$ occurs in S_m; $P_{k_j \xi_i}$ the number of times the $(i+1)$-ad $k_j \xi_i$ occurs in S_m. The formula mentioned in the title is as follows:

$$\sum_{{}_j\alpha_{i-1}} [P_{k_j \xi_{i-1}} - P_{k_i \xi_{i-2}} + P_{k_j \xi_{i-3}} - \ \ldots \pm P_{k_j \xi_1}] = (-1)_i [m + P\alpha_i - P_j \alpha_{i-1}],$$

where α_i is any given set, $k_1, k_2, \ldots, k_i$, of K, when k_j is any element of α_i, and ${}_j\alpha_{i-1}$ represents the set of $i-1$ elements obtained by removing k_j from α_i. The summation is extended over all $\xi_{i-1}, \xi_{i-2}, \ldots$ contained in ${}_j\alpha_{i-1}$. This formula may readily be summed from $j = 1$ to $j = i$, yielding a second formula.

The above formula for $i = 3$ gives

$$P_{ij} + P_{ik} + P_{jk} = m + 2\, P_{ijk}.$$

In any system S_m, in which all pairs occur the same number of times, all triples will also occur the same number of times. This theorem is readily generalized. Application of the formula is made to certain problems of arrangement —in particular to certain problems connected with a certain type of whist tournaments.

(3) Professor Carver's paper called attention to the difficulty caused for the student by the two different notions of the trigonometric functions, as functions of *angles* and functions of *numbers*. It was suggested that the student might

be helped (1) in trigonometry, by the early introduction, and *frequent use in problems*, of the radian unit; (2) in analytic geometry, by insistence upon the use of the radian unit in the plotting of such curves as $y = \sin x$ and $\rho = a\theta$; (3) in calculus, by a clear presentation of the notion of trigonometric functions of *numbers* just before the development of their derivatives.

(4) Mr. Whittemore's paper, which was an account of the course in Artillery Firing given the past year at Yale University as a three-hour course for a semester, has already appeared in this MONTHLY. It was a very timely address and in its full form offers a great aid to those who have been planning similar courses.

(5) A first step toward a combined course for freshmen was taken at the University of Rochester by Professor Gale in the year 1907–1908, when it was decided to emphasize the graphical significance of the functions occurring in the subjects taught, namely, solid geometry, trigonometry and advanced algebra. Soon thereafter considerable interweaving of these subjects was done. In 1913–1914 the calculus was introduced and a combined course, developed by Professors Gale and Watkeys, has been taught for the past four years.

The central theme of the year's work is the study of the elementary functions, algebraic and transcendental. An attempt has been made to keep in very close contact with applications of mathematics, and to attach greater importance to ideas and the development of the power to think than to purely manipulative processes. An analysis of the properties of functions and graphs consists of relations between a single function and its graph, and of the relations between various pairs of functions and their graphs. These principles may be introduced in connection with simple algebraic functions, and utilized in the presentation of the transcendental functions. Interpretation of the graphs of important functions furnishes the means for organizing and remembering various properties of the functions.

Among the applications, emphasis is laid on the determination of a function whose table of values agrees reasonably well with a given table obtained empirically, the general problem being given as a part of the method of discovery in science.

It is believed that the moulding of the subject matter into a coherent year's work gives the student a better comprehension of what mathematics is and a greater facility in its use, and that the freshmen are more interested in their work than formerly.

Among those who took part in the discussion, Professor Richardson maintained that different courses should be given according as the students do or do not expect to continue the study of mathematics, Professor Williams told of his success in combined courses for freshmen, and Professor Olds recalled the interest which was kindled by the introduction of calculus into the freshman year at Amherst College years ago.

(6) Professor Young made a report of the progress being made along the lines of the committee's activities. (See report in the MONTHLY for December, 1917, page 463, for a statement of these.) Miss Blair's report has been postponed in

its printing in order to assimilate with this report certain material which has come in in connection with two or three recent articles on formal discipline and the transfer of training.

The survey in charge of Professor G. W. Evans of courses in algebra in secondary schools is being correlated with the work of committees appointed by the associations affiliated with the Association in this work. Professor Crathorne has made a preliminary report and is now engaged in studying the results of questionnaires.

It seems altogether probable that within the next few months the committee will be able to present a report embodying definite principles, the report to be submitted to the Association and the affiliated bodies for their discussion and promulgation if the report is adopted.

(7) The address of ex-President Cajori has already been printed in *Science*. The removal of residence of Professor Cajori to California rendered his presence at the meeting impossible, and in his absence the paper was read by Professor Olds.

(8) Professor Kennedy showed by clear and excellent charts how he meets the difficulties of the students as they begin the study of descriptive geometry. He uses a set of large scale diagrams carefully drawn on heavy board in various colors, blue print and problem sheets 8″ by $10\frac{1}{2}$″ issued to the pupils, as well as tests on sheets of the same or double size, such tests of varying length being used once a week to secure concentration of effort and a rating of the student's work.

No text is used in the course. A preliminary sheet shows the method of representing geometrical figures by means of three reference planes (frame of reference) and the three working rules of orthographie projection, *e. g.*, vertical and horizontal projections lie on the same vertical line. Blue prints give successively symbols and conventions, simple and clear perspective and orthographic drawings of the elementary theorems of points and lines, planes and traces, lines and planes (*e. g.*, point of intersection of a given line and a given plane), angle between two planes, the gradual approach by rotations to axonometric and isometric projection, a plate of "facts and pitfalls" (correct and incorrect constructions side by side) mimeographed in blue and red respectively.

The large scale wall charts previously mentioned illustrate the gradual development of a problem (*e. g.*, the problem of passing a plane through three given points was shown in four stages, besides a perspective drawing) and afforded an accurate and neat drawing not feasible in quick blackboard sketching. Colors are freely used to distinguish different situations, as for example given, construction, and required lines. Some of the charts have somewhat the nature of dissolving views, in that lines, etc., not longer of importance are drawn more lightly or are omitted in the later stages.

A complete and concise notation is used both in assignment of problems and in the student's analysis of these; thus, the statement "Given a line ab and a point p, to pass a plane M through p perpendicular to ab" is given in the following form:

$$\frac{\overline{ab} \text{ and } p}{M \to p \perp \overline{ab}}$$

The student is expected to note plainly the definite steps in his solution by the use of this notation and other means; for example, an arrow denotes the direction in which he revolves a line. Moreover this notation in connection with given lines, points, etc., mimeographed on problem sheets enables him quickly to attack the assigned problems each day. So far as shades and shadows are taught, a very few charts are used instead of a text, all oral explanation being obviated by the use on these charts of the abbreviated notation already familiar to the class.

Papers are marked with respect to accuracy, clearness and workmanship. Only limited stress is laid on originality since this is apt to develop into eccentricity. Ink is used sparingly and always where it will do the most good.

Professor Kennedy showed a plate embodying a ten minute exercise, a mid-year examination, and a plate of four problems comprising an "inverted test," *i. e.*, four problems were drawn incorrectly, the student being given fifteen minutes to pick out the errors, indicate and correct these.

Dean Randall, who was to have discussed this paper, was unable to be present by reason of the unexpected pressure of duties in inaugurating the new war courses. Professor Willson showed a copy of Monge's *Géométrie Descriptive*, published in 1799 during the French Revolution and dated according to the calendar invented for the new regime by the author and a collaborator; since the use of this calendar ceased soon thereafter this is a very unique book, being the only descriptive geometry and one of the very few books which bear this system of dating.

Meeting of the Council of the Association.

At the meeting of the Council on Friday evening, the following fourteen persons, on applications duly certified, were elected to individual membership:

L. Bianchi. Prof., Univ. of Pisa, Pisa, Italy.
Ugo Broggi, Ph.D. (Göttingen). Prof., Buenos Aires and La Plata Univs., Buenos Aires, Arg.
C. S. Cox, A.M. (Vanderbilt). Prof., Southern Coll., Birmingham, Ala.
P. J. da Cunha. Prof., Univ. of Lisbon, Lisbon, Portugal.
Federigo Enriques. Prof., Univ. of Bologna, Bologna, Italy.
G. A. Gilbert. Prof. of physics and math., St. Ignatius Univ., San Francisco, Cal.
F. D. Murnaghan, Ph.D. (Johns Hopkins). Instr., Johns Hopkins Univ., Baltimore, Md.
H. R. Park, A.B. (Southern Univ.) Teacher, Jun. Coll., Riverside, Cal.
Arthur Pelletier. Prof. of higher alg., École Polytechnique, Montreal, Can.
Salvatore Pincherle. Prof., Univ. of Bologna, Bologna, Italy.
Susan M. Rambo, A.M. (Smith). Asst. prof., Smith Coll., Northampton, Mass.
L. H. Rice, Ph.B. (Syracuse). Instr., Tufts Coll., Tufts College, Mass.

A. V. RICHARDSON, M.A. (Cambridge). Lecturer, Bishop's Coll., Lennoxville, Quebec, Can.

A. G. WEBSTER, Ph.D. (Harvard), Sc.D. (Tufts), LL.D (Hobart). Prof. of physics, Clark Univ., Worcester, Mass.

Several of those elected come as the result of a movement for enlisting foreign members, which has been carried on the past year or more by the Committee on Membership, Professor E. R. Hedrick, chairman.

In this place should be announced a list of twenty-seven persons and three institutions elected to membership by mail vote of the Council in July, 1918:

To individual membership:

R. A. ARMS, Ph.D. (Penna.). Prof., Juniata Coll., Huntingdon, Pa.

R. N. ASHMUN, A.M. (Univ. of Wash.). Computer, Internat. Boundary Commission, Washington, D. C.

H. G. AVERS, A.B. (George Washington Univ.). Coast and Geodetic Survey, Washington, D. C.

H. E. BURTON, M.S. (Iowa). Asst., U. S. Naval Observatory, Washington, D. C.

GENEVIEVE E. COFFREY, A.M. (Univ. of Wash.). Instr. in science, High School, Mackay, Ida.

C. H. GINGRICH, Ph.D. (Chicago). Prof., Carleton Coll., Northfield, Minn.

J. M. HACKLER, Ph.B. (Chicago). Chair of math., Northeastern State Normal, Tahlequah, Okla.

EMMA E. HANTHORN, A.B. (Nebraska). Instr., State Normal, Kearney, Neb.

A. C. HICKMOTT, B.S. (Dartmouth). Statistician, Conn. Gen. Life Ins. Co., Hartford, Conn.

W. G. HUBERT, Sc.D. (New York Univ.). Instr., Coll. of City of New York, New York, N. Y.

C. B. HUGINS, B.S. in M.E. (Carnegie Inst.). Wilkinsburg, Pa.

LAURA M. LUNDIN, B.S. (Mass. Inst. of Tech.). Asst. prof., Wheaton Coll., Norton, Mass.

A. G. MONTGOMERY, A.B. (West Virginia). Instr., Concord St. Normal, Athens, W. Va.

MARY S. MOONEY, A.M. (Bellevue College). Dean of women and prof., Henderson-Brown Coll., Arkadelphia, Ark.

A. L. ONDRAK, A.B. (St. Procopius). Secy., St. Procopius Coll., Lisle, Ill.

R. E. POWELL, E.E. (Ga. Sch. of Tech.). Industr. High Sch., Columbus, Ga.

J. M. RANKIN, A.B. (Maryville Coll.). Instr., Coll. of Idaho, Caldwell, Ida.

PERCIVAL ROBERTSON, Ph.B. (Yale). Instr., The Principia, St. Louis, Mo.

H. E. RUSSELL, A.M. (Wesleyan), Sc.D. (Denver). Prof., Univ. of Denver, Denver, Col.

G. E. F. SHERWOOD, A.M. (Harvard). Asso. prof., Col. Sch. of Mines, Golden, Col.

R. K. STEWARD, C.E. (Maine). Prof. of drawing and design, Mich. Agric. Coll., East Lansing, Mich.

ELLA A. M. THORP, A.B. (Minn.). Asst. instr., Univ. of Minn., Minneapolis, Minn.
G. M. V. TRYON, Fenton, Mich.
G. P. UNSELD, A.B. (Colorado). Grad. stud., Univ. of Col., Westminster, Col.
C. B. WATTS, A.B. (Indiana). Asst., U. S. Naval Observatory, Washington, D. C.
J. J. WIDMAYER, Jr., M.S. (St. John's Coll.). Structural designer, Navy Dept., Washington, D. C.
C. C. WYLIE, A.M. (Missouri). U. S. Naval Observatory, Washington, D. C.

To institutional membership:

SOUTHWESTERN COLLEGE, Winfield, Kan.
COLLEGE OF ST. THOMAS, St. Paul, Minn.
NEW MEXICO NORMAL UNIVERSITY, East Las Vegas, N. M.

The Council transacted further business in connection with the MONTHLY and with the annual meeting of the Association. In view of the developments at the Dartmouth meeting it seemed inevitable and of the greatest importance that the subject of mathematics courses for the period of the war should form the predominating feature of the December program. Whether the Association should meet in affiliation with the American Mathematical Society, whether in affiliation with the American Association for the Advancement of Science at Baltimore, whether the Association will best serve the interests of its members by holding one meeting or, instead of this, enable more members to be within reach of the meetings by having duplicate programs in the East and in the West, were the subjects of earnest discussion. By the time this report appears it will doubtless have been decided by the full participation of the Council as to what place is wisest.

W. D. CAIRNS, *Secretary-Treasurer.*

MATHEMATICAL ENCYCLOPEDIC DICTIONARY.

By G. A. MILLER, University of Illinois.

The following preliminary article is intended to serve as a basis for discussions relating to the nature and the extent of the major articles in the proposed mathematical dictionary. It aims to explain the terms *group* and *group theory* and to define a few of the most important terms which are related thereto. The latter terms should probably appear in their regular alphabetical places with references to the words *group* or *group theory* for their special meanings in this connection. In some cases this special meaning could not be made clear without such preliminary general developments as are here presented.

The object has been to give only such information as is within the range of the first-year graduate student, since the proposed dictionary should clearly not aim

at completeness but should confine itself to rendering very efficient service to certain classes of students. It is evidently a difficult matter to select those elements of a large subject which a student may be supposed to know at a certain stage of his development, in view of the great differences in interest and preparation. On the other hand, brief expositions which aim to present only what is most fundamental have a peculiar charm even for the mature students.

While the chairman of the "Dictionary Committee" of the Association was consulted as regards the desirability of publishing a preliminary article, he did not see the present article itself in manuscript, so that the committee is in no way responsible for its style or content. [See note on page 428. ED.] The writer believes that the major mathematical terms should appear in five languages, English, French, German, Italian and Spanish, and hence the term group appears in these languages, in order, at the beginning of the article.

Group (grōōp), *groupe, Gruppe, gruppo, grupo.* This term is frequently used in mathematics with its non-technical meaning, denoting a collection composed of a finite or of an infinite number of elements; *e. g.*, a group of terms, a group of points, a group of waves, group insurance, etc. The most general technical mathematical meaning of the word group is obtained by restricting its non-technical meaning by imposing the condition that each of the elements of the collection can be combined with itself and also with every other element of the collection. Moreover, the result obtained by such a combination must also be an element of the collection. This most general technical meaning of the term group includes all its other technical meanings as well as that of domain of rationality. It appears frequently in the geometrical literature, but it occurs also in the literature on analysis. Cf. *Encyclopédie des Sciences Mathématiques*, tome 1, volume 2, page 243.

The most common additional restriction imposed on the elements of a group is that the *inverse* of each element shall also be contained in the collection; i. e., each element of the collection can be combined with at least one element of the collection so as to obtain the *identity*. Among the most common additional restrictions are the following: when the elements are combined, or *multiplied together*, the associative law holds, and if any two of the symbols in the equation

$$xy = z$$

are replaced by elements of the collection the equation has always one and only one root in the collection.

All these restrictions are usually imposed on the elements of a finite collection representing a mathematical group, but frequent exceptions appear when the collection contains an infinite number of elements. Cf. Lie, *Theorie der Transformationsgruppen*, volume 1, page 163, where an infinite group which does not involve the inverse of any one of its elements is discussed. Some writers contend that even an infinite collection does not represent a group unless its elements satisfy all the conditions of combination noted above. Cf. A. Loewy, *Archiv der Mathematik und Physik*, volume 9, (1905), page 105, where it is stated that the definition given on page 218, volume 1, *Encyklopädie der Mathematischen Wissenschaften*, is incorrect as regards infinite groups, because it is satisfied by a collection of elements which does not include the inverses of its elements. A group in the most restrictive sense of the term has been called *ordinary* group by L. Autonne, Paris, *Comptes Rendus*, vol. 143 (1906), p. 671.

E. Galois (1811–1832) seems to have first used the term group with a technical mathematical meaning. In fact, Galois and the other writers before the middle of the nineteenth century practically confined their group theory studies to finite groups whose elements are represented by substitutions, and every finite collection of substitutions satisfies the most restrictive definition of group noted above provided it satisfies the most general technical definition. Difficulties as regards the most desirable definitions of the technical term group began to appear when A. Cayley inaugurated the study of abstract groups (1854), and C. Jordan exhibited the wide usefulness of infinite groups (1868). These difficulties cannot be regarded as solved at the present time.

In a broad way groups have been divided into four categories, as follows: Finite discontinuous, infinite discontinuous, finite continuous and infinite continuous. The first of these categories of groups began to be studied during the latter half of the eighteenth century in connection with the solution of algebraic equations in one unknown, and the terminology thus developed was largely transferred to similar concepts arising in connection with the study of the other categories. Among the important concepts common to all of these

categories are the following: subgroup, invariant subgroup, conjugate groups and quotient group. It may be noted that some groups which are continuous in the terminology of S. Lie (1842–1899) have been called discontinuous by H. Taber, *Bulletin of the American Mathematical Society*, vol. 6 (1900), p. 202.

An illustration of a group in the most general mathematical sense is furnished by the natural numbers when they are combined by addition or multiplication, or by both of these operations separately. When the positive and negative integers together with 0 are combined by addition there results a group in the most restricted sense. Both of these are *infinite discontinuous* groups. An instance of a *finite discontinuous* group is furnished by the 24 movements of space which transform a cube into itself, while the totality of the movements of space which transform a point into itself constitutes an illustration of a *finite continuous* group. The elements of a continuous group are not denumerable. Such a group is composed of one or several families of transformations each depending upon a finite number of parameters.

An infinite continuous group contains an infinite number of parameters, or one or more parametric functions. The theory of infinite continuous groups has been less developed than that of finite continuous groups. In all cases except perhaps that of finite discontinuous groups there is as yet no uniformity of usage as regards the essence of the restrictions to be imposed on the most general technical definition of a group although the restrictions imposed by S. Lie have been very widely adopted in works on continuous groups. It should be emphasized that a collection alone is not a technical group. There must be also some law of combination.

The concept involved in the most general technical definition of the term group is that of a system of elements which is *closed* as regards one or more than one method of combining these elements. This concept is as old as mathematics itself since it appears in the fundamental operations of arithmetic and it also underlies Euclidean geometry. Cf. H. Poincaré, *The Monist*, volume 9, 1899, where it is stated, on page 31, that without the notion of group there would be no geometry. In fact, it appears probable that the successive extensions of the number concept, so as to include fractions, and negative, irrational and complex numbers, was largely due to the fact that the most general technical notion of group is ingrained in us and has influenced our intellectual development. Notwithstanding the great importance of this general technical notion and the need of a special name for it, it is too general to serve as a basis of a special abstract theory at the present stage of mathematical development.

Efforts to formulate an abstract definition of group, sufficiently special to serve as a basis for an autonomous theory, were inauguarated after the most general technical notion of group had been extensively applied to a set of concrete elements (substitutions) which intrinsically obey the associative but not the commutative law of combination. The theory of substitutions, more than the theory of numbers, thus furnished the model for abstract definitions of a group, which were first clearly formulated by H. Weber (1882), and G. Frobenius (1887). Somewhat earlier (1870) L. Kronecker had formulated such a definition for an abstract abelian group. Among later formulations of such definitions we would refer especially to those which appeared in the *Transactions of the American Mathematical Society* and are due to E. H. Moore (1902, 1904, 1905), L. E. Dickson (1905), E. V. Huntington (1903, 1905, 1906). Cf. W. A. Hurwitz, *Annals of Mathematics* (1906–1907), page 94.

An explicit abstract formulation of a definition of group in the most general technical sense, including both continuous and discontinuous groups, was given by S. Lie in 1871, *Forhandlinger Videnskabs-Selskabet*, edited in 1872, page 243; but in the development of his theory of continuous groups Lie imposed additional restrictions so that his groups can be defined by means of differential equations. In the preface to volume 3 of his *Theorie der Transformationsgruppen* (1888–1893), which aims to give a pure abstract theory of finite continuous groups, he directed attention (p. 17) to the fact that F. Klein used the term continuous group in a more general sense than that adopted by Lie.

Among the other terms used for group are the following: *Permutation* (Ruffini), *system of conjugate substitutions* or *conjugate system* (Cauchy), *closed system* (Lie and Klein in their earliest publication only). The term employed by Cauchy has been extensively used by others, especially by French writers. In particular, it is used instead of group in the *Cours d'algèbre supérieure* by J. A. Serret, being retained in the sixth edition (1910). When a part of the elements of a group constitute a group the latter is called a *subgroup* or a *divisor* of the former.

When the elements of a group are symbols of operation having no intrinsic properties the group is said to be *abstract;* when these symbols have intrinsic properties the group is said to be *concrete*. Abstract groups are sometimes called *general* groups. It is therefore necessary to distinguish between general groups and groups satisfying the most general technical definition of this term. When the elements of a group combine according to the commutative law the group is said to be *commutative* or *abelian*. When the elements of a group represent geometric concepts the group is called a *geometric* group. Groups which are not geometric are usually called *algebraic* or *analytic*. When the elements of a group are represented

by substitutions it is called a *substitution* group.

If s_1 and s_2 represent two elements of a group G the element $s_1^{-1}\, s_2\, s_1$ is called the *transform* of s_2 with respect to s_1. When s_2 remains fixed while s_1 is replaced successively by all the elements of G there results a totality of elements known as the *complete set of conjugates* of s_2 under G. The elements of this set when they are combined in every possible manner generate a group which includes the complete set of conjugates of all its elements under G. When this group is not identical with G it is said to be an *invariant* or *self-conjugate subgroup* of G. An invariant subgroup is characterized by the fact that it includes the complete sets of conjugates of each of its elements, and when a group does not contain any such subgroup besides the identity it is called a *simple* group. A group which is not simple is said to be *composite*. Invariant subgroups are also called *normal divisors, proper divisors, monotypic, self-conjugate, etc.*

When all the elements of a group H are transformed by an element s of H, or of a larger group G in which H appears as a subgroup, there results a group H' known as the *conjugate* of H with respect to s. It results directly that H is, in turn, the conjugate of H' with respect to the inverse of s, or s^{-1}. When s represents successively all the elements of G there results a *complete set of conjugates of* H under G. A necessary and sufficient condition that H is invariant under G is that all of these conjugates are identically equal to each other. All the elements of G can be divided with respect to H into sets (known as *co-sets*), and when H is invariant these sets combine as units and constitute a group called the *quotient group* of G with respect to H, which is denoted by G/H. A quotient group is also called a *factor* group, or a *complementary* group.

The elements of an abstract group are often called *operators* or *operations*. When the number of these operators is finite it is called the *order* of the group. On the other hand, the order of a finite continuous group is the number of its arbitrary parameters. A substitution group on n letters is said to be of *degree* n. Some writers use order for degree and vice versa. A substitution group is said to be *regular* when its order is equal to its degree and every substitution besides the identity involves all the letters.

When each letter of a substitution group is replaced by a particular letter in some substitution of the group it is called a *transitive* group. A regular group is necessarily transitive. When a substitution group is not transitive it is said to be *intransitive*. A transitive substitution group whose letters can be divided into sets such that every substitution of the group transforms all the letters of each of these sets either among themselves or into those of another one of these sets is called *imprimitive* or *non-primitive*. All transitive groups which are not imprimitive are said to be *primitive*. While the term primitive is positive it is usually defined negatively and the converse is true as regards the term imprimitive.

It is important to note that in the theory of continuous groups the intransitive groups are classed with the imprimitive groups while this is not done in the case of finite substitution groups. A substitution group which involves all the possible substitutions on its letters is called *symmetric*. The symmetric group of degree n is of order $n!$ The substitution group composed of exactly half the substitutions of a symmetric group is called *alternating*. When a substitution group has the property that it replaces a sub-set of r of the letters by every possible such set of its letters it is said to be *r-times* or *r-fold* transitive. The alternating group of degree n is $(n-2)$-fold transitive while the symmetric group of this degree may be said to be either $(n-1)$-fold or n-fold transitive.

There are groups which are composed of a finite number of families of continuous transformations. For instance, the movements of the plane which transform a point into itself and are represented by the following two sets of equations:

$$x' = x\cos\theta - y\sin\theta,$$

$$y' = x\sin\theta + y\cos\theta,$$

and

$$x' = x\cos\theta + y\sin\theta,$$

$$y' = x\sin\theta - y\cos\theta.$$

These two families of continuous transformations constitute a group, but one cannot pass continuously from a transformation of the former type to one of the latter since the determinants of the former are equal to unity while those of the latter are equal to -1. Such groups are sometimes called *mixed* or *complex* groups.

There is still a considerable lack of uniformity as regards the use of terms in group theory. For instance, the term *principal group* (Hauptgruppe) was used by G. Frobenius (*Crelle*, vol. 86, 1879, p. 219) to represent the group formed by the identical element, while F. Klein employed the same term for the continuous mixed group of movements whose invariants constitute elementary geometry. Similarly, the term *anharmonic* group has been employed for two distinct groups; viz., for a group of order 6 which obeys the same laws of combination as the symmetric group of degree 3, and hence is said to be *simply isomorphic* with this group (Pascal's *Repertorium*, vol. 1, 1910, p. 238), and also for the non-cyclic group of order 4, commonly known as the *four-group* (Capelli, *Istituzioni di analisi algebrica*, 1909, p. 111).

Group theory. The systematic development of theorems relating to properties of

groups is known as group theory. Among the various branches of this science the theory of substitution groups is the oldest, having been founded by A. L. Cauchy, about 1845, and first embodied in the form of a separate treatise by C. Jordan, *Traité des substitutions et équations algébriques*, 1870.

The theory of substitutions contains two large branches. The older of these is sometimes called the theory of permutation groups, and is based on the possible interchanges of letters, while the other branch is involved in the theory of linear transformations, and is commonly known as the theory of linear substitution groups.

The theory of finite abstract groups is intimately connected with the two theories of substitution groups just noted and was first embodied in the form of a separate treatise by W. Burnside, *Theory of groups of finite order*, 1897; second edition with greater emphasis on linear groups, 1911. These three theories are sometimes referred to as *algebraic group theory*. There is, however, no clear line of distinction between algebraic group theory and the group theories of analysis and geometry.

The group theory of analysis may also be divided into three large branches, viz., theory of finite continuous groups, theory of infinite continuous groups, and theory of groups of automorphic functions. The first of these theories was first developed in a systematic manner by S. Lie, *Theorie der Transformationsgruppen*, three large volumes, while the last was treated in volume 1 of *Automorphe Functionen* by R. Fricke and F. Klein, 1897. No systematic treatise on the general theory of infinite continuous groups has yet been published.

Geometric group theory is based on the group theories of algebra and analysis. In geometry the group concept has entered more widely into the various developments than in algebra or in analysis. Among the treatises devoting considerable space to geometric groups we may mention Klein's *Einleitung in die höhere Geometrie*, II, 1893, and Lie's *Geometrie der Berührungstransformationen*, 1896.

C. Alasia prepared a general bibliography on group theory, which was published in volumes 18–22 of *Rivista di fisica matematica e scienze naturali*, Pavia. A bibliography relating to finite groups together with many historical data may be found in the *Constructive development of group theory* by B. S. Easton, 1902. Among the treatises on the theory of groups which were not noted above are the following: E. Netto, *Substitutionentheorie*, 1882; translated into Italian by G. Battaglini, 1885, and into English by F. N. Cole, 1892; S. Lie and G. Scheffers, *Vorlesungen über Differentialgleichungen*, 1891, and *Vorlesungen über kontinuierliche Gruppen*, 1893; G. Vivanti, *Teoria dei gruppi di transformazioni*, 1898; translated into French by A. Boulanger, 1904; L. Bianchi, *Lezioni sulla theoria dei gruppi di sostituzioni*, 1900; L. E. Dickson, *Linear Groups*, 1901; J. E. Campbell, *Theory of Continuous Groups*, 1903; J. A. de Séguier, *Groupes Abstraits*, 1904; G. Fubini, *Teoria dei gruppi discontinui e delle funzioni automorfe*, 1908; H. Hilton, *Finite Groups*, 1908; E. Netto *Gruppen- und Substitutionentheorie*, 1908; J. A. de Séguier, *Groupes de Substitutions*, 1912; Miller, Blichfeldt and Dickson, *Theory and Applications of Finite Groups*, 1916; H. F. Blichfeldt, *Finite Collineation Groups*, 1917.

FUNDAMENTALS IN THE MATHEMATICS OF INVESTMENT.

By EDWARD LEWIS DODD, University of Texas.

§ 1. Introduction.

In most books on the mathematics of investment there is a wealth of formula somewhat forbidding to the casual reader, however necessary it may be to the accountant or actuary. It is the object of this paper to present in rather compact form some of the fundamentals of the subject, with a few general formulas of wide application.

§ 2. Interest and Discount.

The mathematics of investment deals with the *increment* of *value*. If P units of value—say P dollars—at one moment of time are worth or conceived to be worth S units at a later moment, the increment $S - P$ is called the *interest*[1]

[1] For the general theory, it is immaterial whether the change of value is brought about by a loan or by a series of commercial transactions, indeed, whether the increment is positive or negative.

(on P), and this same difference is called the *discount* (on S) for the period of time determined by the given moments. The ratios

$$r = \frac{S - P}{P}, \qquad u = \frac{S - P}{S} \tag{1}$$

are respectively the *rate* of *interest* and the *rate* of *discount* for the period.

Discount is also called *interest in advance*; S is called the *amount* of P, and P the *present value* or *present worth* of S; P is often called the *principal* or *capital*.

If each dollar of P increases by r, the total increase is $S - P$, the given increase. The interest-rate for a period, then, may just as well be defined as the increase of *one* (dollar) during the period. In place of a dollar, any unit of money or value may be used.

Likewise, if from each dollar of S, the value at the end of the interval, the same deduction u is made, the total deduction is $S - P$, the difference between the ultimate and initial value of the money in question. Thus u may be defined as the discount on *one* (dollar) for the period.

The sum of money $1/(1 + r)$ at the beginning of the period becomes *one* at the end of the period. The former is, then, the *present value* or *present worth* of the latter, and may be designated by w. Then

$$\frac{1}{1 + r} = w = 1 - u. \tag{2}$$

Illustration.—A man borrows \$100 for one year at a discount of 4%—or at 4% interest in advance. In this case, he actually receives \$96 and must pay \$100 at the end of the year. Thus, \$1 due in one year is worth \$.96 now; $w = .96$. The discount on \$1 is \$.04; $u = .04$. The interest is \$4 on \$96, the interest rate is about $4\frac{1}{6}\%$; $r = .0416+$.

§ 3. Derived Interest Rates.

Let r be the interest rate for each of n consecutive periods[1] of time. Then an initial P (dollars) becomes $P(1 + r)$ at the end of the first period, $P(1 + r)^2$ at the end of the second period, $\cdots$, $P(1 + r)^n$ at the end of the nth period. For the sake of simplicity, the P may be dropped. The amount of *one* for the entire period or *term*—formed by joining the n consecutive periods—is

$$1 + r_n = (1 + r)^n,$$

where, by the definition of § 2, r_n is the *interest* on *one* for the entire period or term.

In conformity with this, the amount of *one* for t periods,[2] where t is any

[1] The periods need not be of equal length. The interest charged for February may be the same as for March. So far as the theory goes, there is no reason why the periods should be even approximately equal.

[2] Even this does not require the periods to be of equal length. Moments of time are merely to be in one-to-one correspondence with real numbers, the later of two moments to be associated with the number algebraically greater.

positive real number, is defined to be

$$1 + r_t = (1 + r)^t. \tag{3}$$

The interest on *one* for the entire period is then

$$r_t = (1 + r)^t - 1. \tag{4}$$

If w_t and u_t refer to this new period, it follows from (2) and (3) that

$$\frac{1}{1 + r_t} = 1 - u_t = w_t = w^t = (1 - u)^t. \tag{5}$$

Illustration.—Many banks pay a "*nominal* 4%" *convertible semi-annually.* This means that they pay 2% for 6 months. Then \$100 becomes \$102 at the end of 6 months, and this \$102 becomes \$104.04 at the end of the next 6 months. This makes \$4.04 the interest on \$100 for one year, and the interest rate for the year is $4\frac{1}{25}\%$. This is in conformity with (4) where the original period is 6 mo., $r = .02$, $t = 2$, $r_t = .0404$.

§ 4. Compound Interest vs. Simple Interest.

If in Equation (4), the Binomial Theorem is used and only two terms retained, the result is simple interest on *one* (dollar). The error of the approximation, using simple interest for compound, is the sum of the terms after the second in the binomial expansion.

If P at one moment is worth Q at another moment, then P and Q will be said to be *equivalent* to each other. Thus, equivalence involves the notion of time as well as of value.

A fundamental property of *compound interest* is the following:

Two sums of money each equivalent to a third sum of money are equivalent to each other.

Thus P at one time is equivalent to $P(1 + r)^t$ after the lapse of the time t. This in turn is equivalent to $P(1 + r)^{t+t'}$ after the further lapse of time t'. But the latter is also equivalent to the original principal P after the lapse of the time $t + t'$.

Thus the initial and the ultimate value are each equivalent to the middle value, and they are equivalent to each other.

But, if *simple interest* is used, two sums of money each equivalent to a third are not equivalent to each other.

For major financial computations, simple interest would be absurd—although it often gives a permissible approximation for a fraction of a year.

Thus, the amount must be an exponential function of the time; it can not be a linear function of the time.

The amount is, indeed, a linear function of the principal. Thus often we may ignore the principal at first, and merely use it as a multiplier as the concluding step in a problem.

Illustration.—If at 5% *simple* interest, \$100 is loaned for 4 years, it becomes \$120; and if this is reloaned for 6 years, it becomes \$156. Whereas, if the \$100 were loaned for 10 years straight the amount would be only \$150.

But if \$100 is kept continuously at strict 5% *compound* interest for 10 years, the amount will be exactly \$100 $(1.05)^{10}$, even if the money changes hands a dozen times.

§ 5. Perpetuities.

A *perpetuity*[1] is an infinite series of values, associated with moments of time which extend indefinitely into the future. These moments are usually thought of as the end moments of the periods into which they divide time. When it is desirable to associate the values—usually called *payments*—with the initial moments of the periods, the perpetuity is called a *perpetuity-due*, or, often, an *immediate perpetuity*. Unless otherwise sta ed, the payments of the perpetuity are to be taken as all equal; indeed, frequently it is understood that each payment is a payment of *one* (dollar).

If r is the interest rate for each period, the *present value of a perpetuity* of *one* per period, payable at the end of each period forever is

$$b_\infty = \frac{1}{r}; \tag{6}$$

and the *present value of a perpetuity-due* of *one* per period, payable at the beginning of each period forever is

$$B_\infty = \frac{1}{u}. \tag{7}$$

For $1/r$ will yield as interest *one* at the end of each period forever; and $1/u$ will yield as interest in advance *one* at the beginning of each period forever. The interest is to be withdrawn as soon as it falls due.

Illustration.—A building must be reconstructed at the end of every 25 years, at an expense of \$10,000. What sum of money put out at 4% compound interest will pay for the renewals forever? Let r_{25} be the interest rate for the period of 25 years. Then from (6) the required endowment is

$$10{,}000\left(\frac{1}{r_{25}}\right) = \frac{10{,}000}{r_1}\left(\frac{r_1}{r_{25}}\right) = 250{,}000\ (.0240120), = \$6003,$$

as found by using a monetary table.

A perpetuity deferred t periods is a series of perpetual payments, the first payment to be made after the lapse of $1 + t$ periods,—thus the first payment is made t periods later than it would normally be made. This t may be any positive real number,—indeed t may be negative, if the *forborne perpetuity*, to be considered presently, is counted as a special case of a deferred annuity. It follows from (5) and (6) that

[1] If \$4 is to be collected as interest on \$100 at the end of each year forever, this is called a perpetuity.

$$ {}_t|\, b_\infty = w^t \left(\frac{1}{r}\right) \tag{8}$$

is the present value of *one* per period forever, the first payment to be made after the lapse of $1 + t$ periods.

Likewise, for the *perpetuity-due deferred t periods,*

$$ {}_t|\, B_\infty = w^t \left(\frac{1}{u}\right) \tag{9}$$

is the present value of *one* per period forever, the first payment to be made after the lapse of t periods.

Now $(1 + r)^t b_\infty$ at any given moment is equivalent to b_∞ for that moment ea lier by t periods, and hence is the "present value" of a perpetuity whose first payment was made $t - 1$ periods earlier. Thus, for this *forborne perpetuity,*

$$ {}_{-t}|\, b_\infty = (1 + r)^t \left(\frac{1}{r}\right) \tag{10}$$

is the sum of the "amounts" of the earlier payments of *one* per period, begun $t - 1$ periods earlier, and the "present values" of the later payments of *one* per period ad infinitum.

Likewise, for the *perpetuity-due forborne t periods,*

$$ {}_{-t}|\, B_\infty = (1 + r)^t \left(\frac{1}{u}\right) \tag{11}$$

is the sum of the amounts of the earlier payments of *one* per period, begun t periods earlier, and the present values of the later payments of *one* per period in regular continuation forever.

§ 6. Annuities.

An *annuity* is a series of periodic payments. The payments are usually equal and limited in number.

A perpetuity may be considered as an annuity with an infinite number of payments. *And an annuity may be considered as the difference between two perpetuities starting at different times.*

The value of an annuity may be required at any time. But usually its value is required (1) at the time of the first payment, or (2) one period before this payment is made, or (3) at the time of the last payment, or (4) one period later than the last payment. Taking n as the number of payments, these four values, in order, are

$$ B_n = \frac{1}{u} - w^n \left(\frac{1}{u}\right) = \frac{u_n}{u}, \tag{12}$$

$$ b_n = \frac{1}{r} - w^n \left(\frac{1}{r}\right) = \frac{u_n}{r}, \tag{13}$$

$$c_n = (1+r)^n\left(\frac{1}{r}\right) - \frac{1}{r} = \frac{r_n}{r}, \tag{14}$$

$$C_n = (1+r)^n\left(\frac{1}{u}\right) - \frac{1}{u} = \frac{r_n}{u}, \tag{15}$$

where each payment is *one*, as is seen by referring to Equations (3)–(11).

These four formulas can be proven without reference to perpetuities. To prove (14), note that an initial *one* (dollar) is worth r per period, payable at the end of each period for n periods, together with *one* at the end of the nth period—this terminal *one* is principal returned. But an initial *one* is also worth $(1+r)^n$ at the end of the nth period. Hence r per period for n periods is worth $(1+r)^n - 1$ at the end of the nth period. Thus *one* per period for n periods is worth $\{(1+r)^n - 1\}/r = r_n/r$, at the end of the nth period.

The most common proof of these formulas involves the summing of a geometric progression.

Illustration.—If a man deposits, in a bank that pays 4%, \$100 at the end of each year for 25 years, the accumulation to his credit at the end of the 25 years will be \$4,164.59, as found from a table based upon (14).

The terms *deferred* and *forborne* are applied to annuities in the same way as to perpetuities. Thus, the present value of an annuity *deferred* t periods is $w^t b_n$.

B_n and C_n are respectively the "present value" and "accumulation" of an *annuity-due*,—here a payment is made at the *beginning* of each interval.

In (12)–(15), the r and u are respectively the interest-rate and discount-rate for the *period between payments*. If an interest rate is given for some other period—say for a period of length t—then by (3) and (2) these formulas (12)–(15) may be transformed to involve r_t explicitly.

The following interesting and useful relations may be easily proved algebraically or arithmetically:

$$b_n = w^n c_n, \qquad B_n = w^n C_n,$$

$$\frac{1}{b_n} = \frac{1}{c_n} + r, \qquad \frac{1}{B_n} = \frac{1}{C_n} + u.$$

§ 7. Varying Annuities and Perpetuities.

It has been found that $1/u$ will furnish *one* at the beginning of each period forever. Required the capital that will furnish *one* at the beginning of the first period, *two* at the beginning of the second period, and so on, increasing forever. These payments will be furnished if a perpetuity-due is started at the beginning of each period. The capital required for this perpetuity-due of perpetuities-due is

$$(IB)_\infty = \frac{1}{u^2}. \tag{17}$$

The formula for a perpetuity-due of perpetuities is

$$(Ib)_{\infty} = \frac{1}{u} \cdot \frac{1}{r} = \frac{1}{r^2} + \frac{1}{r}. \tag{18}$$

This yields *one* at the *end* of the first period, *two* at the *end* of the second period, and so on, increasing forever.

An *increasing* annuity or perpetuity is one in which the successive payments are in order, *one, two, three,* etc., as in counting. Thus, an increasing annuity of n payments is an increasing perpetuity minus an increasing perpetuity started n periods later, minus also n times a perpetuity of *one* started likewise at the latter moment.

A *varying* annuity or perpetuity may have payments forming an arithmetic progression of second or higher order. A reader interested in such annuities is referred to "The Institute of Actuaries' Text Book, Part I," pages 40–48.

§ 8. Complete Annuities.

Suppose that the interest r on *one* (dollar) is collected at the end of each of m consecutive periods. If the loan is continued for the fraction t of a period, and the "*face*" of this loan—viz. *one*—is then collected, interest for that fraction t of a period is also due, to the extent of r_t, as per (4). Now the interest payments of r each form an annuity, and the single payment of r_t is said to *complete* this annuity. Likewise, if $1/r$ is loaned, an annuity of *one* per period results, which is *completed* by the payment of r_t/r.

Let the whole term of the loan be n periods. Then $n = m + t$. Now an initial $1/r$ has at the end of n periods the value $(1 + r)^n/r$. Thus, reasoning as in § 6, *one* per period for m periods, followed by the completing payment of r_t/r, is worth at the end of n periods,

$$\frac{1}{r}(1 + r)^n - \frac{1}{r} = \frac{r_n}{r},$$

as in (14). Formula (14) has thus been made valid for all positive real values of n provided that the completing payment is r_t/r when n is not integral. This completing payment approaches one when t approaches one, as obviously it should.

Indeed, formulas (12)–(15) are valid for all positive values of n if the completing payment of r_t/r is made at the end of the n periods.

Likewise they hold if the set of m regular payments of *one* each is preceded by an initial completing payment of u_t/u, made earlier than the first regular payment by the fraction t of a period.

Indeed, these formulas will remain valid if any set of k consecutive regular payments is replaced by a single payment of u_k/u at the beginning of the term of the special set, or by a single payment of r_k/r at the end of the term of the special set.

If the t above is a fraction of a small period of time, such as a year, a fair approximation for r_t/r, or indeed for u_t/u, will be t itself, in most cases—as easily seen by using the Binomial Theorem.

Illustration.—On a loan of \$2,500 at 4%, interest payments of \$100 have been collected regularly for a certain number of years. The lender wishes the loan of \$2,500 paid 6 months after the last interest payment has been made. In practice \$50, the simple interest on the \$2,500 for 6 months would be collected. But this is not the equitable interest. \$100 payable at the end of a year is equivalent to \$49.50 payable at the end of 6 months and \$49.50 at the end of the year. The interest on \$49.50 for 6 months makes up the extra dollar. An annuity of \$100 per year is completed (exactly) by \$49.50 at a moment 6 months later than a regular payment. This illustrates the fact that *for a fraction of one period, simple interest is greater than compound interest,* so that a money lender can well afford to substitute simple interest for compound interest for a fraction of the specified interest period.

§ 9. Common Notation.

To gain generality, certain commonly accepted symbols have been avoided in this paper thus far.

When the year is taken as the unit of time, the usual symbols are as follows:

$$r = i, \qquad u = d, \qquad w = v,$$

$$b_n = a_{\overline{n}|}, \qquad c_n = s_{\overline{n}|}, \qquad B_n = \mathbf{a}_{\overline{n}|}, \qquad C_n = \mathbf{s}_{\overline{n}|}.$$

Thus

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \qquad s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

The corresponding rate of interest for one one-mth of a year is j_m/m; and the corresponding rate of discount is f_m/m. Here j_m is called the "*nominal rate of interest,*" and f_m the "*nominal rate of discount.*"

The equations

$$1 + i = \left(1 + \frac{j_m}{m}\right)^m = v^{-1} = (1 - d)^{-1} = \left(1 - \frac{f_m}{m}\right)^{-m} = e^{\delta}$$

connect the most fundamental of these quantities where the force of interest

$$\delta = \log_e (1 + i) = \lim_{m=\infty} j_m = \lim_{m=\infty} f_m,$$

with i constant.

§ 10. Conclusion.

No attempt has been made to display all the formulas most frequently used, or to introduce the reader to the many fascinating applications in connection with premiums on bonds, amortization schedules, wearing values of machinery, sinking funds, etc.

Few students of mathematics have any conception of the beauty or difficulty of certain problems arising in actual business transactions.

But an intense arithmetic appreciation of the important relations underlying annuities and perpetuities will go far toward equipping a student to solve problems of this kind.

BOOK REVIEW.

SEND ALL COMMUNICATIONS ABOUT BOOKS TO W. H. BUSSEY, University of Minnesota.

Introduction to Mathematical Statistics. By CARL J. WEST, Ph.D., Assistant Professor of Mathematics, Ohio State University. R. G. Adams and Co., Columbus, 1918.

In spite of the wide divergence between the original sources and purposes of mathematical statistics, the present development of the subject seems to be along two main lines—that is, either it presents statistical information consisting usually of numerical measurements in a form easily and quickly interpreted by the eye, or it derives and applies formulas for the purpose of measuring various phenomena presented by the measurements.

The first line of development calls for very elementary mathematical knowledge and has been well treated by several authors, especially Brinton. The second line of development has been treated in its various phases in scientific journals and a few English and German books; it involves mathematical principles ranging from the most elementary to the most abstruse but was never presented by an American statistician in approximately complete form until Dr. West's *Introduction to Mathematical Statistics* appeared. The usefulness of the German books in this country is impaired by the language while all the other foreign books on the subject are too voluminous to be used as textbooks in American colleges.

As may be expected, Dr. West's book gives a treatment of both lines of development and, considering the great amount of literature of the second kind in various scientific journals, shows good judgment in the selection of important principles. No doubt, Dr. West's book will be widely used—especially as a textbook in colleges—with few rivals for many years to come, although any author of a statistical textbook would merit considerable praise even though he did little more than encourage wider study and help to standardize methods.

The reviewer agrees with Dr. West in most of his introductory statements but insists that a knowledge of at least the calculus is indispensable to a full understanding of the principles involved in the second line of development mentioned above, especially the work of Pearson and his followers, some of which is treated by Dr. West.

The printers show a lack of experience with scientific books or else the proof was not carefully read; the alignment is poor in places—for example, letters are out of line in eighteen different places on page 18—and typographical errors are fairly frequent.

A few paragraphs are poorly written. For example, the last paragraph on page 131 is surely not in the form intended. Also, the expression *Since* a_1 *and* a_2 *are not integers*, in the last paragraph on page 135, seems to be out of place. The repetition of the formula for K on page 138 seems unnecessary—it is given on the preceding page. The formulas for S_3, S_4 and S_5 on page 98 are almost too badly mixed and the expression for $1/y \cdot dy/dx$ in terms of the moments is almost too poorly aligned to be unravelled by the student. But the trouble in both cases is typographical.

The latter part of the book is a little too condensed, especially the part of the appendix devoted to the Pearson types of frequency curves, which cannot be used except for purposes of reference unless supplemented by explanations and illustrations by an experienced instructor. Reference should have been made to the types of curves recently developed by Pearson as well as to the interesting abacus given in Pearson's Tables to be used in distinguishing the various types.

The treatment of the various kinds of averages is good although the reviewer believes that the weighted arithmetic average should be regarded as a special form of the ordinary arithmetic average—where the various conceptions of *weight* take the place of *frequencies*.

Correlation theory is treated at great length, as it deserves, although the important correlation surface and its equation, with its important relation to the correlation coefficient, is unfortunately omitted in order to avoid more advanced mathematical principles. The distinction between the correlation coefficient and the correlation ratio is well explained.

Most of the errors found in the book are trivial and of the kind which it is almost impossible to avoid in a book dealing with a virgin field. On the whole, Dr. West's book promises to fill well the long felt want for a textbook on mathematical statistics and no longer need instructors depend solely upon their own notes and lectures. The fact that the book is well provided with good examples should make it especially successful as a text-book.

C. H. FORSYTH.

DARTMOUTH COLLEGE.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Missouri.

It is to be understood that problems proposed for solution or solutions of problems which have been proposed in the MONTHLY are welcomed from all readers, whether subscribers or not. Single copies containing these problems or solutions will be sent to those contributing, provided their addresses are known to the Secretary of the Association.

2727. Proposed by H. J. WOODALL, Stockport, England.

Having given $2^k \equiv +k \pmod{p}$ and ξ the haupt-exponent of 2 for mod p (ξ is the least power of 2 whose residue, for modulus p, is plus unity) solve $x \cdot 2^x + 1 \equiv 0 \pmod{p}$.

Also, as regards the inverse problem, what do we know about the factors of $N \equiv b \cdot 2^b + 1$, when b is a known positive integer, e. g., $b = 141$.

2728. Proposed by NORMAN ANNING, Somewhere in France.

A material triangle of uniform density and thickness is of such a shape that when suspended from the vertices in succession, the lower sides have slopes of 1 : 1, 1½ : 1, and 3 : 1. Construct the triangle given that the shortest side is 10 inches.

By definition, an a : 1 slope makes an angle with the vertical whose tangent is a.

2729. Proposed by N. P. PANDYA, Sojitra, India.

Solve in integers $x^3 + 3y^4 = z^2$.

2730. Proposed by W. E. HEAL, Washington, D. C.

If $z^n = x^n + y^n$, where x, y, z, n are integers, and n prime, prove that

$$\{z^{(n-1)/2}[(z-x)x^{(n-1)/2} + (z+y)y^{(n-1)/2}]\}^2 + \{x^{(n-1)/2}[2x^{(n+1)/2} + y^{(n-1)/2}(x-y)]\}^2$$
$$= \{z^{(n-1)/2}[(z+x)x^{(n-1)/2} + (z-y)y^{(n-1)/2}]\}^2 + \{y^{(n-1)/2}[2y^{(n+1)/2} - x^{(n-1)/2}(x-y)]\}^2$$
$$= [z^{n+1} + (x^{(n+1)/2} - y^{(n+1)/2})^2] \times [z^{n-1} + (x^{(n-1)/2} + y^{(n-1)/2})^2].$$

Also prove that $z^{n+1} + (x^{(n+1)/2} - y^{(n+1)/2})^2$ and $z^{n-1} + (x^{(n-1)/2} + y^{(n-1)/2})^2$ have no common factors except the divisors of $z^{n-1}(z+x)(z+y)$.

2731. Proposed by JAMES K. WHITTEMORE, Yale University.

A bowl is in the form of a paraboloid of revolution. If for a given volume the surface is a minimum, prove that the ratio of the diameter of the top to the depth is approximately 1.86.

SOLUTIONS OF PROBLEMS.

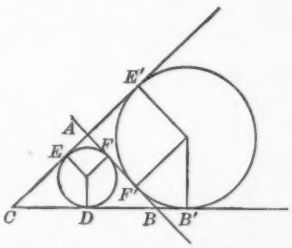
2660. Proposed by JOSEPH E. ROWE, State College, Pa.

Prove that the distance measured along the side of a triangle, from the point of contact with the inscribed and escribed circle, is equal to the side of the triangle between the two circles.

SOLUTION BY GEORGE F. WILDER, Brooklyn, N. Y.

Let ABC be the triangle and s its semi-perimeter. From the equalities $AE = AF$, $CE = CD$, and $BD = BF$, we have $s = AE + CD + BD$, or $s = AE + a$, since $CD + BD = a$. Hence, $AE = s - a$, (1). Also, since $AE' = AF'$ and $BF' = BD'$, $CE' + CD' = 2s$. Hence, $CD' = s$, since $CE' = CD'$. Hence, $BD' = CD' - CB = s - a$, (2). From (1) and (2) we have $AE = BD'$.

Now $EE' = EA + E'A = BD' + AF'$, since $AE = BD'$ and $AE' = AF'$, and hence $EE' = BF' + AF' = AB$.



Similar and various other solutions were received from the following contributors: J. V. BALCH, HORACE OLSON, PAUL CAPRON, S. W. REEVES, C. P. SOUSLEY, J. F. CONNELLY, ELIJAH SWIFT, and the PROPOSER.

This problem was proposed in an examination test in Cooper Union, New York, and solutions similar to the one above were received from G. J. HARRIS, H. F. MATHUSSEN, I. J. FAJANS, C. J. SCHMITT, H. GLADSTONE, MORRIS DEVORKEN, ANGELO J. SAJIN, DAVID TENNENBAUM, JACOB GREISMANN, I. MILLENKY,

MAX SILVERMAN, G. VANDERBOON, J. LEBEDEFF, P. WEINBERG, MOSE KAHN, MORRIS GROSS, PETER PUMO, HYMAN MILBERG, S. L. RADER, O. C. SIMONSEN, ABRAHAM PLETMAN, FRANK PATLI, M. J. MONTFERRANTE, JOSEPH MENT, MORRIS BUNSIS, SAMUEL COHEN, and CLIFFORD STRAIN.

2668. Proposed by B. F. FINKEL, Drury College.

Show that

$$v = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

where a is the radius of a droplet, σ its density, η the viscosity of the air and v the velocity under gravity g. *Stokes's law.*

SOLUTION BY ALBERT R. NAUER, St. Louis, Missouri.

Stokes finds analytically a formula determining the resistance of a fluid medium to the movement of a sphere through it,

$$F = \tfrac{4}{3}\pi\rho an\sqrt{-1}[af_1'(a) + 2f_2(a)]e^{\sqrt{-1}nt}, \tag{I}$$

Stokes's *Mathematical and Physical Papers*, Vol. III, p. 33. From a previous development (p. 28) is given

$$f_1''(r) - \frac{2}{r^2}f_1(r) = 0, \quad \text{and} \quad f_2''(r) - \frac{2}{r^2}f_2(r) - m^2f_2(r) = 0. \tag{II}$$

Then the equations of condition become on putting $f(r)$ for $f_1(r) + f_2(r)$, and $r = a$,

$$f'(a) = ac, \qquad f(a) = \tfrac{1}{3}a^2c, \tag{1}$$

$$f'(b) = 0, \qquad f(b) = 0. \tag{2}$$

The integration of the differential equations (II) has no difficulties, the first one is of a well-known form and the second one is a Riccati equation solvable in finite terms.

The integrals are

$$f_1(r) = \frac{A}{r} + Br^2 \quad \text{and} \quad f_2(r) = Ce^{-mr}\left(1 + \frac{1}{mr}\right) + De^{mr}\left(1 + \frac{1}{mr}\right).$$

Here $D = 0$, since otherwise the velocity would be infinite at an infinite distance; $B = 0$, otherwise the velocity would be finite when $r = \infty$.

From Eq. (1), we get

$$A = \tfrac{1}{3}a^3c - aCe^{-ma}\left(1 + \frac{1}{ma}\right).$$

Putting $b = \infty$, we get from Eq. (2),

$$\frac{d}{da}\{\text{Equation (1)}\} = 0 = \frac{3a^2c}{2} - aC(1 - m - 1)e^{-ma}, \qquad C = \frac{3ac}{2m}e^{ma},$$

and then

$$A = \tfrac{1}{3}a^3c + \frac{3a^2c}{2m}\left(1 + \frac{1}{ma}\right).$$

Substituting in (I),

$$F = -\tfrac{2}{3}\pi a^2cn\sqrt{-1}\left(1 + \frac{9}{ma} + \frac{9}{m^2a^2}\right)e^{\sqrt{-1}nt}. \tag{III}$$

Substituting for m its value, which is the particular square root of $(n\sqrt{-1})/\mu'$ which has its real part positive, and write V for $ce^{\sqrt{-1}nt}$. Then

$$F = -6\pi\mu'\rho aV, \tag{IV}$$

where $\mu' = \mu/\rho$, called by Stokes, the *index of friction* and ρ is the density of the fluid. Whenever the motion is so slow that the part of the resistance which depends on the square of the velocity may be neglected, we have, supposing V to be the terminal velocity;

$$-F = \tfrac{4}{3}\pi ga^3(\sigma - \rho),$$

where g is the force of gravity, and σ, which is supposed to be greater than ρ, the density and a the radius of the sphere.

Hence,

$$6\pi\mu'\rho aV = \tfrac{4}{3}\pi g a^3(\sigma - \rho),$$

or

$$V = \frac{2}{9}\frac{g}{\mu'}\left(\frac{\sigma}{\rho} - 1\right)a^2 = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

by placing

$$\frac{\sigma - \rho}{\mu\rho} = \frac{\sigma}{\eta}.$$

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

CONCERNING TWO FIFTH-POWER PROBLEMS IN DIOPHANTINE ANALYSIS.

By CYRUS B. HALDEMAN, Ross, Butler County, Ohio.

I. *To resolve*

$$p^5 + q^5 + p^5 + s^5 = t^5 + u^5 + v^5 \tag{1}$$

where p, q, r, s, t, u, and v are rational.

Solution. Let $p = 2x$, $q = y - x$, $r = 2a - b^2 - x$, $s = 2a + b^2 + x$, $t = x + y$, $u = 2a - b^2 + x$ and $v = 2a + b^2 - x$.

Substituting in (1) we get

$$\begin{aligned}(2x)^5 + (y - x)^5 + (2a - b^2 - x)^5 + (2a + b^2 + x)^5 \\ = (x + y)^5 + (2a - b^2 + x)^5 + (2a + b^2 - x).^5\end{aligned} \tag{2}$$

We may write (2):

$$\begin{aligned}(x + y)^5 + (x - y)^5 + [(2a + b^2) - x]^5 - [(2a + b^2) + x]^5 \\ + [(2a - b^2) + x]^5 - [(2a - b^2) - x]^5 - (2x)^5 = 0.\end{aligned} \tag{3}$$

Expanding, regarding each binomial in parenthesis as a single quantity, and adding, we have from (3)

$$\begin{aligned}2x^5 + 20x^3y^2 + 10xy^4 - 2x^5 - 20x^3(2a + b^2)^2 - 10x(2a + b^2)^4 \\ + 2x^5 + 20x^3(2a - b^2)^2 + 10x(2a - b^2)^4 - 32x^5 = 0.\end{aligned} \tag{4}$$

Equation (4) reduces to

$$2x^2y^2 + y^4 - 16ab^2x^2 - 64a^3b^2 - 16ab^6 - 3x^4 = 0.$$

Solving for y^2, we get

$$y^2 + x^2 = \sqrt{4x^4 + 16ab^2x^2 + 64a^3b^2 + 16ab^6} = 2x^2 + 4a(b^2 + d^2), \text{ say.}$$

From this we find

$$x^2 = \frac{4a^2b^2 - (b^2 + d^2)^2 a + b^6}{d^2},$$

and

$$y^2 = \frac{4a^2b^2 - (b^2 + d^2)^2 a + b^6 + 4d^2(b^2 + d^2)a}{d^2}.$$

To rationalize the expressions for x and y, let

$$d^2x^2 = 4a^2b^2 - (b^2 + d^2)^2\, a + b^6 = (n - 2ab)^2, \tag{5}$$

and

$$d^2y^2 = 4a^2b^2 - (b^2 + d^2)^2\, a + b^6 + 4d^2\,(b^2 + d^2)\, a = (n - 2ab - Z)^2. \tag{6}$$

From (5) we get

$$a = \frac{n^2 - b^6}{4bn - (b^2 + d^2)^2},$$

and from (6), since the square of $n - 2ab$ is assumed equal to the first four terms of the left-hand member, we have

$$a = \frac{2nZ - Z^2}{4bZ - 4d^2\,(b^2 + d^2)}.$$

Equating the values of a and clearing of denominators, we get

$$4b^6d^2\,(b^2 + d^2) - 4b^7Z - 4d^2\,(b^2 + d^2)n^2$$
$$= Z^2\,(b^2 + d^2)^2 - 4bZ^2\,n + 4bZn^2 - 2Z\,(b^2 + d^2)^2\,n.$$

Place $4bZn^2 = -\,4d^2\,(b^2 + d^2)n^2$ and solve for

$$Z = -\,\frac{d^2\,(b^2 + d^2)}{b};$$

then

$$n = \frac{4b^7Z - 4b^6d^2\,(b^2 + d^2) + Z^2\,(b^2 + d^2)^2}{4bZ^2 + 2Z\,(b^2 + d^2)^2}.$$

Substituting the value of Z in n gives

$$n = \frac{d^2\,(b^2 + d^2)^3 - 8b^8}{2b\,(d^4 - b^4)}.$$

The value of Z and the value of n in terms of b and d, substituted in either of the above values of a gives

$$a = \frac{d^4(b^2 + d^2)^2 - 4b^8}{4b^2(d^4 - b^4)}.$$

From (5) and (6) we have

$$x = \frac{n - 2ab}{d},$$

and

$$y = \frac{n - 2ab - Z}{d}.$$

The values of a, n and Z substituted in these expressions for x, and y, give

$$x = \frac{bd^2(b^2+d^2)^2 - 4b^7}{2d\,(d^4-b^4)} \quad \text{and} \quad y = \frac{bd^2(b^2+d^2)^2 - 4b^7}{2d(d^4-b^4)} + \frac{d(b^2+d^2)}{b}.$$

From the values of a, x and y we obtain

$$p = \frac{bd^2(b^2+d^2)^2 - 4b^7}{d(d^4-b^4)}, \qquad q = \frac{d(b^2+d^2)}{b},$$

$$r = \frac{2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)}{2b^2d(d^4-b^4)},$$

$$s = \frac{2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)}{2b^2d\,(d^4-b^4)},$$

$$t = \frac{d^4(b^2+d^2) - 4b^8}{bd(d^4-b^4)},$$

$$u = \frac{2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)}{2b^2d(d^4-b^4)},$$

$$v = \frac{2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)}{2b^2d(d^4-b^4)}.$$

By substitution of the expressions just found for p, q, r, s, t, u and v in (1); and after clearing of denominators, we have the identity

$$\begin{aligned}
&[2b^3d^2(b^2+d^2)^2 - 8b^9]^5 + [2bd^2(b^2+d^2)(d^4-b^4]^5 \\
&\qquad + [2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)]^5 \\
&\qquad + [2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)]^5 \\
&= [2bd^4\,(b^2+d^2)^2 - 8b^9]^5 \\
&\qquad + [2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)]^5 \\
&\qquad + [2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)]^5
\end{aligned}$$

Take $b = 1$, $d = 2$, and we have, after dividing by 12^5,

$$16^5 + 50^5 + 53^5 + 79^5 = 63^5 + 66^5 + 69^5.$$

II. *To resolve the equality*

$$\begin{aligned}
&p^5 + q^5 + r^5 + s^5 + t^5 + u^5 + v^5 + w^5 \\
&\qquad = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \ldots + (2g_n)^5,
\end{aligned} \tag{1}$$

where p, q, r, s, t, u, v, w, g_1, g_2, g_3, . . . , g_n are rational.

Solution. Let $p = a + b + c$, $q = a - b - c$, $r = b - a - c$, $s = c - a - b$, $t = d + e + f$, $u = d - e - f$, $v = e - d - f$, $w = f - d - e$.

Then, by substitution in (1), we have

$$(a+b+c)^5 + (a-b-c)^5 + (b-a-c)^5 + (c-a-b)^5 + (d+e+f)^5 + (d-e-f)^5 + (e-d-f)^5 + (f-d-e)^5 = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \ldots + (2g_n)^5. \quad (2)$$

For convenience, we may write (2):

$$[a+(b+c)]^5 + [a-(b+c)]^5 + [(b-c)-a]^5 - [(b-c)+a]^5 + [d+(e+f)]^5 + [d-(e+f)]^5 + [(e-f)-d]^5 - [(e-f)+d]^5 = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \ldots + (2g_n)^5. \quad (3)$$

Expanding, regarding each binomial in parenthesis as a single quantity and adding, we get from (3)

$$2a^5 + 20a^3(b+c)^2 + 10a(b+c)^4 - 2a^5 - 20a^3(b-c)^2 - 10a(b-c)^4 + 2d^5 + 20d^3(e+f)^2 + 10d(e+f)^4 - 2d^5 - 20d^3(e-f)^2 - 10d(e-f)^4 = 32\,(g_1^5 + g_2^5 + g_3^5 + \ldots + g_n^5). \quad (4)$$

Equation (4) reduces to

$$10a^3[(b+c)^2 - (b-c)^2] + 5a[(b+c)^4 - (b-c)^4] + 10d^3[(e+f)^2 - (e-f)^2] + 5d\,[(e+f)^4 - (e-f)^4] = 16(g_1^5 + g_2^5 + g_3^5 + \ldots + g_n^5);$$

and by further expansion and reduction, we have

$$5(a^3bc + ab^3c + abc^3 + d^3ef + de^3f + def^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5). \quad (5)$$

Let $d = a$ and (5) becomes

$$5a^3\,(bc + k^3\,ef) + 5a\,(b^3c + bc^3 + ke^3f + kef^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5).$$

Place $bc + k^3ef = 0$ and have

$$5a(b^3c + bc^3 + ke^3f + kef^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5).$$

From these equations we obtain

$$b = -\frac{k^3ef}{c}, \qquad a = \frac{2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5)}{5(b^3c + bc^3 + ke^3f + kef^3)},$$

which give the requirements.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

CLUB ACTIVITIES.

DENISON MATHEMATICS CLUB, Denison University, Granville, Ohio.

This Club came into existence through well attended meetings of an informal nature in 1915 shortly after Professor Forbes B. Wiley went to Denison University as head of the department of mathematics. In January, 1916, a constitution was approved and a club formally organized "to bring before its members matters of interest in mathematics that are not regularly discussed in courses offered in the curriculum." Any college student or member of the faculty who desires to join the club is eligible to do so. The average attendance at meetings last year was about 25 as against 30 the year before.

Officers 1917–18: President, Grace Jefferson '18; vice-president, Ruth Phillips '19; secretary-treasurer, Sterling Abell '20; assistant secretary-treasurer, Lawrence Curl '20. The program committee consists of these officers together with the head of the department.

October 3, 1916: Election of officers for the year.

November 7: The purposes of the club explained by the president, and various short topics presented to illustrate to new-comers some of the possibilities of such an organization.

November 21: "Problems in the mathematics of astronomy" by Professor Wiley.

December 19: "History of the Denison Mathematics Club"[1] by Marie Tilbe '17; "Integral curves" by Professor Anna B. Peckham.

January 23, 1917: "Mathematical societies and journals" by Professor Wiley; attention was drawn to problems in this MONTHLY.

February 20: Solution of problems proposed at the previous meeting by Lawrence Curl '20 and Virgil Traxler '19; "Spherical projection" by Harlan C. Reynolds '17.

March 6: "The possible use of parallel axes in the plane of intersecting axes of coördinates" by Professor Wiley.

March 20: "The concept infinity" by Professor Paul Biefeld of the astronomy department.

April 17: "Hyperbolic functions" by Professor Wiley.

May 1: Officers for 1917–18 elected.

September 25: "The probability function" by Professor Wiley.

October 23: "The fourth dimension" discussed by Sterling Abell '20, George Read '18, Clifford Marshall '18 and Charles T. Bumer '19.

November 6: "Theorems of elementary number theory" by Professor Wiley.

November 20: "Inversion" by Professor Peckham.

[1] Referring, presumably, to meetings before the formal organization.

December 18: "Constructions with compasses alone and with ruler alone" by George T. Street Jr., instructor in mathematics.

January 22, 1918: "The algebra of number pairs" by Grace Jefferson '18.

February 19: "The possible use of parallel axes in the plane of intersecting axes of coördinates" (continued)[1] by Professor Wiley.

March 5: "The different methods of defining the trigonometric functions and the identification of the functions so defined" (primarily for freshmen) by Professor Wiley.

March 19: "A problem in invariants" by Mildred Hunt '08.

April 16: "Introduction to the study of groups" by Professor Wiley.

May 14: Election of officers for 1918–19.

"During the past year our Club has felt increasingly the loss of strong members who have entered the national service."

UNDERGRADUATE MATHEMATICS CLUB, University of Illinois, Urbana, Ill.

The first mathematics club at the University of Illinois was organized during the year 1899–1900. Members of the faculty, graduate students and undergraduates constituted the membership. At the bi-weekly meetings for several years the topics discussed were suitable for undergraduates but gradually the club became an experiment station for trying out papers to be presented later before the American Mathematical Society. As a result in 1909 the Club was divided into two sections: (1) the graduate section with its tri-weekly meetings devoted mostly to the presentation of doctor's dissertations and original papers by faculty members; and (2) the undergraduate section which met monthly, for about an hour, "for the consideration of questions of general mathematical interest and the solution of problems." During several years the average attendance at meetings of this latter section was 30–35. While the management of the section was largely in the hands of graduate students a majority of the speakers were undergraduates. This arrangement did not seem to give entire satisfaction and still further bifurcation occurred in the autumn of 1917.

There are, then, now three organizations: I. Mathematics Club—Graduate Section, all members of the department staff and all graduate students being *eo ipso* members. The meetings are open to all interested and are devoted exclusively to reports on research work done in the department and to occasional reports on scientific meetings. II. The Mathematical Round Table, consisting of all graduate students in mathematics (and a few selected seniors) but not open to men with the Ph.D. degree. At the weekly meetings the average attendance is 12–15. Each member presents two papers a year on some semi-advanced topic not taught in the regular courses. Visitors are not encouraged. III. The Undergraduate Mathematics Club from which the department staff, as well as graduate students, are excluded. The members consist of juniors and seniors majoring in mathematics, and freshmen and sophomores of excellent stand-

[1] *Cf.* this MONTHLY, 1918, page 255.

ing in mathematics courses. All undergraduates are welcome as visitors. The average attendance is 8–12 from a membership of 36. Officers, 1917–18: President, Ruth Andrews '18; vice-president, Mary D. Craigmile '18; secretary, Harry W. Penhallow '19; program committee: the president, George Williams '18, and Dr. Aubrey J. Kempner (faculty adviser).

Apart from the solving of problems less difficult than those in the MONTHLY the programs of the Undergraduate Mathematics Club for last year were as follows.

January 23, 1918: "On paper folding" by Ruth Anderson '18.
February 6: "The value of high school mathematics" by Winifred White '18.
February 20: "Mathematical puzzles" by Fannie McMurray '19.
March 13: "On Rolle's Theorem" by George Williams '18.
April 3: "Constructions with the double ruler" by Mary Craigmile '18.
April 17: "The Theorem of Pythagoras" by Irene Doyle '19.
May 1: "The fundamental theorem of algebra" by C. T. G. Ching '18.
May 15: "On algebraic numbers" by Jesse E. Wilkins '18.

The programs of the Mathematics Club—Undergraduate Section—in 1915–16 were as follows.

November 7, 1915: "Who's who in mathematics in America" by Nathan C. Grimes, assistant in mathematics.
December 5: "Construction possibilities" by Katherine Lackey '16.
January 9, 1916: "Fourier Series and the harmonic analyzer" by Paul L. Bayley Gr.
February 13: "Nomography" by Joe Langueville Gr.
March 13: "Origin of calculus" by Erma Elliott Gr.
April 10: "Graphic solution of equations" by Albert E. Babbitt Gr.
May 9: "Mathematical models" by Mauritz Hedlund Gr.

THE MATHEMATICAL CLUB OF THE KANSAS STATE AGRICULTURAL COLLEGE, Manhattan, Kansas.

This club was organized in September, 1913 "to stimulate larger interest in mathematics on the part of students." Any student of collegiate rank was eligible for membership, which totalled 75 in 1916–17; the average attendance at meetings was about 20. Professor Benjamin L. Remick the head of the mathematics department acted as "officer in charge;" there were no student officers. "For various local reasons such as decrease in enrollment due to the war and the desire of several members of the department to engage in special mathematical work for themselves, it was decided not to carry on the usual club activities during 1917–18."

The programs for the first four years of the Club's existence were printed annually. They are reproduced below.

October 25, 1913: "Organizations and journals for the study of mathematics in America" by Professor Remick; "History of π" by Professor Alfred E. White.
November 8: "Euclidean constructions" by Professor William H. Andrews; "The history of logarithms" by Arthur Fehn, instructor.

November 22: "The teaching of secondary algebra" by Ina E. Holroyd, assistant; "The algebraic treatment of the evolute of a conic" by Professor Harrison E. Porter.

December 6: "History of a few proofs of the Pythagorem Theorem" by Charles H. Clevenger, instructor; "The place of limits in geometry" by Daisy D. Zeininger, instructor.

January 17, 1914: (*a*) "Note on problem 411, AMERICAN MATHEMATICAL MONTHLY,"[1] (*b*) "The perfect magic square for 1914" by Professor Remick; "The tendencies in modern mathematics" by Professor William T. Stratton.

January 31: "Concerning regular polyhedra" by Professor William H. Andrews; "The development of irrational numbers" by Professor White.

February 14: "The development of the decimal system" by Mr. Fehn; "Calculating machines" by Professor Porter.

February 28: "A mathematical notation applied to a few problems" by Mr. Clevenger; "The place of mathematics in the education of women and girls"[2] by Miss Holyroyd.

March 14: "Mathematics as a universal study" by Professor Stratton; "What is the laboratory method in the study of mathematics?" by Miss Zeininger.

April 11: "Non-Euclidean geometry" by Professor Remick; "Some fundamental ideas in mathematics" by Professor Andrews.

April 25: "The theory of duality" by Professor Porter; "The mathematical properties of maps" by Professor White.

May 9: "The history of trigonometry" by Mr. Fehn and Miss Zeininger; "Mathematics in English public schools" by Professor Stratton.

May 23: "The problem of failures" by Miss Holyroyd; "Coördinate systems" by Mr. Clevenger.

November 7: "Recent movements in mathematics" by Professor Remick; "Mathematical paradoxes" by Elliott Ranney '16; "History of the duplication of the cube" by Roy W. Haege '18.

November 21: "A problem in partial payments" by Professor Porter; "Graphical railroad time tables" by Gabe A. Sellers '17; "Plato as a mathematician" by Edith L. Alsop '16.

December 5: "Number bases among the primitive races" by Professor Stratton; "Mathematics at Kansas State Agricultural College from a student's standpoint" by William A. Lathrop '15; "History of algebraic symbolism" by Dilts S. McHugh '18.

January 16, 1915: "History of angular measurement" by Professor White; "History of the trisection of an angle" by Jefferson H. Flora '17; "Magic square for 1915" by Rufus S. Kirk '17.

January 30: "Short methods in the four fundamental operations" by Lee R.

[1] Vol. 20, p. 136: "*ABCD* is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line. (Unsolved in *Educational Times*.)" Cf. *Proceedings of the Edinburgh Mathematical Society*, Vol. 28, pp. 152–178, 1909–10.

[2] Published in *School Science and Mathematics*, June, 1914.

Light '15; "Curves that trisect an angle" by Earl E. Swenson '17; "Origin of number symbols" by Caroline R. Packard '17.

February 13: "Number concept and generalization in algebra" by Mr. Fehn; "Mathematical games" by Zeno C. Rechel '18; "Elementary mathematics in evening schools" by Helen Mitchell '18.

February 27: "The mathematics of mineralogy" by Lyle M. Dean, assistant; "Life and works of Newton" by Louis R. Parkinson '16; "History of arithmetic in the United States" by Mae V. Hildebrand Sp.

March 13: "Correlation of mathematics and physics" by Eustace V. Floyd, assistant professor of physics; "History of the quadrature of the circle" by Robert F. Mirick '16; "Systems of quadratic equations" by James A. Hull '17.

April 10: "The logical and the psychological in geometry" by Miss Zeininger; "The controversy between Leibnitz and Newton" by Charles A. Willis '16; "Mystic properties of numbers" by Charles H. Zimmerman '16.

April 24: "Mathematical symmetry in nature"[1] by Miss Holyroyd; "Construction of logarithmic and trigonometric tables" by Donald D. Hughes '18; "Life and works of Descartes" by Russel H. Oliver.

May 1: "Thought versus rule in mathematics" by Joseph I. Kirkpatrick, assistant in veterinary medicine; "Euclid and his geometry" by Carl D. Hultgren '17; "Hypatia, last of the Greeks" by Sarah K. Kimport '18; "Geometry for engineers" by Leroy N. Miller '18.

May 15: "Historical solution of the quadratic equation" by Professor Andrews; "Relation of mathematics to statistics" by Lester Tubbs '17; "Geometry and construction work" by Basil A. Greene Sp.

November 6: "The place of mathematics in the study of heredity" by Edward N. Wentworth, professor of animal breeding.

November 20: "The function idea in and outside of mathematics" by Professor Remick; "The relation of mathematics to wireless telegraphy" by Harold M. McClelland '16; "Mathematics in verse" by Helen Mitchell '18.

December 4: "Generalizing some theorems in elementary geometry" by Professor Andrews; "The United States coast and geodetic survey" by Hubert A. Dawson '19; "The life and work of Poincaré" by Harry L. Robinson '18.

January 15, 1916: "The application of the prismatoid formula to the solids of elementary geometry" by Professor White; "History of arithmetic in the United States" by Gordon W. Hamilton '19; "The Pythagorean brotherhood" by Elizabeth A. Cotton '19.

January 29: Address by John O. Hamilton, professor of physics; "Zero and infinity" by Elliott Ranney '16.

February 12: "Means of measuring mathematical ability in students" by Professor Stratton; "Graphical solution of quadratic equations having complex

[1] In this connection, reference may be given to the somewhat unscientific work by S. Colman and C. A. Coan entitled *Nature's Harmonic Unity. A Treatise on its Relation to Proportional Form*, New York, Putnam, 1912.

roots" by Frank M. Sisson '18; "Proof and history of the fundamental theorem of algebra" by Otto B. Githens '17.

February 26: "The mathematics of investment" by Professor Porter; "The mathematics of chemistry" by Herbert H. King, assistant professor of chemistry.

March 4: "Mathematical symbolism and the economy of thought" by Miss Zeininger; "The solution of the biquadratic equation" by Charles A. Willis '16; "The problem of one cent" by Leroy N. Miller '18.

March 18: "The influence of French mathematics in America" by Miss Holyroyd; Problem discussion by Ivor O. Mall '18; "The solution of the cubic equation" by Andrew M. Harvey '18.

April 8: "History and development of Fermat's last theorem" by Mr. Fehn; "Some noteworthy series for the value of π and their derivation" by Lowell E. Baldwin '18; "Development of imaginary numbers" by Wilbur Lane '19.

April 22: Address by Siebelt L. Simmering, assistant professor of steam and gas engineering; "The cattle problem of Archimedes" by Charles A. Frankenhoff '19; "Two American mathematicians" by Earl V. Kesinger '17.

May 13: "The ancient and the modern treatment of proportion" by Lyle McF. Dean Gr. and assistant; "Mathematics and the science of war" by Jefferson H. Flora '17; Problem discussion by Joseph P. Ball '19.

November 14: "Recent developments in high school mathematics" by Miss Holyroyd.

December 5: "The mathematics of life insurance" by Professor Porter.

January 16, 1917: "The problem of three bodies—history and progress" by Professor Remick; "The construction of logarithmic tables" by Harry Dunham '18.

February 16: "Formulæ for the area of a triangle" by Miss Zeininger; "Some problems in gearing" by Myron R. Bowerman, assistant professor of mechanical drawing and machine design.

March 6: "Elementary geometry of the triangle" by Professor Stratton.

April 10: "Some fundamental ideas in mathematics" by Professor Andrews; "The application of mathematics to civil engineering" by Lowell E. Conrad, professor of civil engineering.

May 8: "Circles of the triangle" by Mr. Fehn; "The relation of mathematics to chemistry" by Herbert H. King, professor of chemistry.

June 5: "Construction of the polygon of seventeen sides" by Professor White; "Graphical solution of the quadratic with complex roots" by Donnelly J. Tarpey '19.

MATHEMATICS CLUB OF THE UNIVERSITY OF MONTANA, Missoula, Montana.

So far as the editor is aware this club was the first one to be founded in 1918. It resulted from a suggestion of its first president. Last semester anyone interested in mathematics could belong to the club but during the present year definite rules for members, as well as a statement of the purposes of the club, are

to be formulated. There were 22 members (3 of the faculty) last year and the average attendance was 18.

Officers 1917–18: President, Gretchen Van Cleve '19; secretary, Adele Maerdian '20; treasurer, Harry Rooney '21. The program committee consists of the president and secretary.

March 11, 1918: Organization of the club.

March 27: Address by Professor Nels J. Lennes, head of the department of mathematics.

April 10: "What other clubs are doing" by Doris Thetge '21; "Russian peasant method of multiplication and other methods" by Adele Maerdian '20.

April 24: "Method of constructing curves on roads and railroads" by Tom Swearingen '19; "Money value of an education" by Bessie Rutledge '20. Election of officers for 1918–19.

May 15: "Mathematics of war" by James Friauf '18.

May 29: "The slide rule" by Radcliffe Beckwith '21.

"At each one of our meetings we have had a 'feed' (a strictly war 'feed,' I assure you,—nothing unpatriotic about us out here in Montana). The young people seem to take interest and pride in getting up a good 'feed,' and they tax us an amount something like fifteen cents to a quarter, each time, to cover the expense. One Sunday we got four automobiles and drove about thirty miles into the country for an afternoon picnic."

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Ill. [*1918*, 132–134].[1]

March 7, 1918: "The history of the teaching of mathematics in the United States" by Mae Campbell '18; "Some mathematical fallacies" by Helen Maloney '18.

March 28: "Bomb throwing" by Frank D. Danielson '18.

April 18: "Gauss's method of quadratures" [subject of Master of Arts thesis] by Theodore Doll '17.

May 2: "Descriptive Geometry" by Milby R. Hammer.

May 18: Election of officers for 1918–19.

MATHEMATICAL CLUB OF ROCKFORD COLLEGE, Rockford, Ill. [*1918*, 187–188].

February 20, 1918: Debate, "Resolved that one year of mathematics should be required in college" by Dorothy Jamison '20, Virginia Schneider '20 (affirmative), and Lila Dole '20, Doris Volland '20 (negative).

March 6: Social meeting.

April 17: "Women in mathematics" by Estle Russell '18.

April 24: Special business meeting.

May 8: "Social meeting open to those freshmen who intend to go on with mathematics and who, therefore, will be able to join the club next year."

[1] This form of abbreviation will be used in the future to indicate earlier pages of this MONTHLY where other items concerning the club may be found.

THE NEWTONIAN SOCIETY OF THE STATE COLLEGE OF WASHINGTON, Pullman, Wash.

This society was founded as The Mathematical Society in November, 1911, but just five years later "after hearing claims made for more than a score of mathematicians, the society voted to name itself in honor of Newton."

"Its purpose is to afford an opportunity for students to form the habit of reading up on assigned topics which, for the most part, do not come up under regular courses. Broadly speaking, it is the aim of this society to lay the foundation for individual investigation and research. The students get some information and much inspiration from attendance upon these meetings. As a rule, one member of the faculty and one student take part in each meeting."

Any member of the faculty and any student interested in mathematics is eligible for membership. The total number of members last year was 14 and the average attendance at meetings was 10. "There are no fees, dues, or other items of expense, in connection with the society."

Officers 1917–18: President, Dorothy Neff '19; vice-president, Dorothea Sorenson '20; reporter (who looks after publicity in the college paper), Flossie Folsom '19; program committee, Professor Charles A. Isaacs, Florence Evans '20 and Rachel Shuman '18.

October 27, 1916: "The continuum" by Professor Isaacs.

November 10: "Life of Euclid" by Rachel Shuman '18; "Life of Archimedes" by Clarence L. Hix, instructor; "Life of Lagrange" by Professor Elmer C. Colpitts; "Life of Newton" by Marie Weldin '17; "Life of Euler" by Frank M. Bryant, instructor; "Life of Descartes" by Professor Isaacs.

November 24: "Mathematical Literature" by Elsie Worthen '20; "Mathematical books in the college library" by Rachel Shuman '18; "Mathematical periodical literature in the college library" by Professor Colpitts.

December 8: "Arithmetic and geometric progressions" by Corrine Barclay '19; "Convergent and divergent series" by Ina Craig '19; "Hypergeometric series" by Mr. Hix.

January 5, 1917: "Methods of Diophantus" by Edith McBride '19; "Arithmetica" by Mr. Bryant.

January 26: "The determinants in algebra" by Dorothy Neff '19; "The determinants in geometry" by Frank Hamelius '18; "Hessians and Jacobians" by Professor Isaacs.

March 2: "Permutations and combinations" by Blanche Lowary '20; "The theory of probability" by Flossie Folsom '19; "Insurance" by Professor Colpitts.

March 16: "Concurrent lines of a plane triangle" by Florence Evans '20; "Properties of a plane triangle" by Marie Weldin '17; "Circles connected with a plane triangle by Mr. Hix.

April 20: "Projectiles" by Rachel Shuman '18; "Gunnery" by Mr. Bryant.

October 18: "Points on a line" by Professor Isaacs; "Properties of a tetrastigm" by Ina Craig '19.

November 15: "Pythagorean astronomy" by Professor Colpitts; "Singular points on a curve" by Edith McBride '19.

December 13: "The ancient and modern abacus" by Mr. Hix.

January 24, 1918: "Symbolic logic" by Professor Isaacs; "Bertrand Russell" by Florence Evans '20.

February 25: "Some problems in modern geometry" by Mr. Bryant and Dorothy Neff '19.

March 21: "What, if any, mathematics should be required for graduates from secondary schools?" by Professor Isaacs; round table discussion by the whole society.

April 1: "Funny figures" by Amy Kelso '21; "Card tricks" by Mr. Hix.

April 11: "The sine law in plane geometry" by Dorothea Sorenson '20; "Addition and subtraction of logarithms" by Professor Colpitts.

May 13: "Arithmetical prodigies" by Flossie Folsom '19; "The nebular and other hypotheses" by Ina Craig '19.

May 16: "The history of the calculus" by Rachel Shuman '18; "Improper integrals" by Elsie Dallas '18.

"The number of programs for the past year was reduced so as not to interfere with the Red Cross work in which many students were engaged."

TOPICS FOR CLUB PROGRAMS.

14. The Cattle Problem of Archimedes.

During the last ten years of his life Gotthold Ephraim Lessing, the German critic and dramatist, occupied himself almost exclusively with the treasures of the library at Wolfenbüttel, Northern Germany, where he was librarian. The results of these researches were embodied in a series of volumes, *Beiträge zur Geschichte und Literatur*, the first being published at Braunschweig in 1773. In this volume first appeared the Greek epigram[1] (in verse form, 44 lines, from the

[1] Page 421 f. The text is followed by Lessing's commentary, a purported solution in Greek, by a scholiast, and a mathematical discussion by Christian Leiste (numerous misprints and errors in connection with the numbers). The same is to be found in the standard edition of Lessing's works: *Sämmtliche Schriften*, herausgegeben von K. Lachmann, besorgt durch F. Munker, Leipzig, Band 12, 1897, pp. 100–107, 110–115. (In this edition the misprints and errors of the 1773 edition have been corrected)—The Greek text of problem and scholium are also given (with Latin translation of the problem) by J. L. Heiberg, *Archimedes opera omnia*, iterum edidit, Vol. 2, Lipsiæ, MDCCCCXIII, pp. 528–534. (See also commentary in J. L. Heiberg, *Quæstiones Archimedeæ*, Hauniae, MDCCLXXIX, pp. 26f., 66f.)—For German translations see: (1) G. F. Nesselmann, *Die Algebra der Griechen*, Berlin, 1842, pp. 481–491; (2) B. Krumbiegel, "Das problema bovinum des Archimedes," *Zeitschrift für Mathematik und Physik*, hist. literar. Abt., 1880, Vol. 25, pp. 121–136. (This comprehensive discussion of earlier work contains German translations of both epigram and scholium. Krumbiegel's paper is followed, pages 153–171, by A. Amthor's mathematical discussion. The two present a masterly presentation of the facts and are fundamental for every student of the problem.)—French translations by Terquem and A. S. C. Vincent in *Bulletin de bibliographie, d'histoire et de biographie mathématiques*, tome 1, 1855, pp. 113–124, 130, 165–173; tome 2, 1856, pp. 39–42.—Italian translation by G. Loria in *Le scienze esatte nell' antica Grecia*, 2a ed., Milano, 1914, pp. 932–939 (Il "problema dei buoi" di Archimede).

The cattle problem was also carefully studied by:

(a) J. Struve and K. L. Struve, Vater und Sohn, *Altes griechisches Epigramm mathematischen*

original[1] discovered by Lessing in the Wolfenbüttel library) headed[2]: "A problem which Archimedes found among (some) epigrams and sent, to be solved by those in Alexandria who occupy themselves with such matters, in his letter to Eratosthenes of Cyrene."

In abbreviated and partly symbolic form the problem is as follows: Compute, O friend, the host of the oxen of the Sun, giving thy mind thereto, if thou hast a share of wisdom, compute the number which once grazed upon the plains of the Sicilian isle Thrinacia,[3] and which were divided according to color into four herds, one milk white, one black, one yellow and one dappled. The number of bulls formed the majority of the animals in each herd and the relations between them were as follows:

1 . . . White bulls (W) $= (1/2 + 1/3)$ black bulls (X) + yellow bulls (Y),
2 . . . Black bulls (X) $= (1/4 + 1/5)$ dappled bulls (Z) + yellow bulls (Y),
3 . . . Dappled bulls (Z) $= (1/6 + 1/7)$ white bulls (W) + yellow bulls (Y),

As to the cows:

4 . . . White cows (w) $= (1/3 + 1/4)$ black herd $(X + x)$,
5 . . . Black cows (x) $= (1/4 + 1/5)$ dappled herd $(Z + z)$,
6 . . . Dappled cows (z) $= (1/5 + 1/6)$ yellow herd $(Y + y)$,
7 . . . Yellow cows (y) $= (1/6 + 1/7)$ white herd $(W + w)$.

Inhalts von Lessing erst einmal zum Drucke befördert, jetgt neu abgedruckt und mathematisch und kritisch behandelt, Altona, 1821 (47 pp.).

(*b*) J. G. Hermann (the German classical scholar and philologist), *De archimedis problemate bovino*, Leipzig, 1828 (12 pp.). (Also in *Godofredi Hermanni Opuscula*, Vol. 4, Lipsiæ, 1831, pp. iii–v, 228–238; contains some notes not in original pamphlet.)

(*c*) P. Tannery: (1) "Sur le problème des boeufs d'Archimède," *Bulletin des sciences mathématiques et astronomiques*, 1881, tome 5, part 1, pp. 25–30 (also in P. Tannery, *Mémoires scientifiques*, tome 1, Toulouse, 1912, pp. 118–123); (2) "L'arithmétique des grecs dans Pappus" (1880), *Mémoires scientifiques*, tome 1, 1912, pp. 103–105.

(*d*) F. O. Hultsch im Pauly-Wissowa's *Real Encyclopädie der Classischen Altertumswissenschaft*, Band 2, Stuttgart, 1896, cols. 531–535, 1110.

Students may find T. L. Heath's somewhat brief discussions more accessible: (1) *The Works of Archimedes*, Cambridge, 1897, pp. xxxiv–xxxv, 319–326. (Also in *Archimedes' Werke mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen* von Sir Thomas L. Heath. Deutsch von F. Kleim. Berlin, 1914, pp. 471–477). (2) *Diophantus of Alexandria, A study in the History of Greek Algebra*, 2d edition, Cambridge, 1910, pp. 11, 12, 121–124, 279.

[1] There is another copy of the epigram in the Bibliothèque nationale, cod. Paris Gr. 2448 saec. XIV, fol 57. Heiberg indicates all the small differences of the codices.

[2] In this connection I have chosen the translation which Krumbiegel and Heath regard as literal, though somewhat unsatisfactory in meaning.

[3] The problem seems to hark back to the twelfth book of Homer's *Odyssey* where the following lines occur: "Next, you will reach the island of Thrinacia where in great numbers feed kine and the sturdy flocks of the Sun" (G. H. Palmer's translation, Boston, 1894, pp. 188–189); or in Pope's translation

"Thence to Trincaria's shore you bend your way,
Where graze thy herd, illustrious source of day!"

As to the name τρινακρία for Sicily, and the θρινακίη of Homer and of the epigram, see *Encyclopaedia Britannica*, article "Sicily," and Homer's *Odyssey* edited by W. W. Merry and J. Riddell, 2d ed., Vol. 1, Oxford, 1886, p. 516.

If thou canst give, O friend, the number of bulls and cows in each herd thou art not unknowing nor unskilled in numbers, but still not yet to be counted among the wise.

Consider, however, the following additional relations between the bulls of the Sun:

8 . . . White bulls (W) + black bulls (X) = a square,
9 . . . Dappled bulls (Z) + yellow bulls (Y) = a triangular number.

When thou hast then computed the totals of the herds, O friend, go forth as conqueror, and rest assured that thou art proved most skilled in the science of numbers.

The first seven equations, in eight unknowns, lead to the solution:

$$W = 10366482\,n,\ X = 7460514\,n,\ Y = 4149387\,n,\ Z = 7358060\,n,$$
$$w = 7206360\,n,\ x = 4893246\,n,\ y = 5439213\,n,\ z = 3515820\,n,$$

where n is any integer. The solution given without explanation by the scholiast corresponds to $n = 80$, and the total number of cattle of the Sun 4,031,125,560. These numbers do not, as he affirms, satisfy conditions 8 and 9.

It is generally accepted that condition 8 is equivalent to $W + X =$ a square *number*; but an ambiquity in the language of the epigram in this connection makes possible the strained interpretation that $W + X =$ a square *figure* or rectangle (i. e. the product of two factors), since a bull is longer than it is broad.

With this latter interpretation of condition 8 the problem was completely solved by J. F. Wurm in 1830,[1] and the total number of cattle of the Sun was then found to be 5,916,837,175,686. If it were desirable to distribute these cattle uniformly over all the dry land of the earth, one animal would come about every 25 square feet.

Amthor attacked the problem on the basis of the more generally accepted reading and this called for the solution of the equation

$$t^2 - 4729494\,u^2 = 1^2$$

After elaborate and arduous work, especially with continued fractions, he arrived at the conclusion that $W = 1598\ \langle 206541 \rangle$, where $\langle 206541 \rangle$ represents the fact that there are 206541 more digits to follow, and that, with the same notation the whole number of cattle $= 7766\ \langle 206541 \rangle$. "It is easy to show that a sphere having the diameter of the milky way, across which light takes ten thousand years to travel, could contain only a part of this great number of animals even if the size of each is that of the smallest bacterium."

There has been much debate as to whether Archimedes really propounded the

[1] Jahn's *Jahrbücher für Philologie und Pädagogik*, Vol. 14, 1830, p. 194 f. Review of Hermann's pamphlet. Amthor gives the solution of Wurm's problem.

[2] This equation has been referred to as a Pellian equation, but there is not the slightest ground for so designating any equation of this form. See *Encyclopédie des sciences mathématiques*, tone 1, Vol. 3, fasc. 1, 1906, p. 27.

cattle problem. In 1880 Krumbiegel gave a pretty complete account of arguments for and against this theory. As a result he concludes: The exact form of the problem is probably later than Archimedes, but as to the problem itself, not only is it very possible, but very probable that it really originated with the celebrated geometer of Syracuse. Such too is the opinion of Tannery, and of Heiberg[1] who is the greatest authority on the texts of Archimedes. Hultsch discusses "a most attractive suggestion" (Heath) that the unmistakable vein of satire in the opening words of the epigram, in the transition from the first to the second parts, and in the last lines, was a shaft directed towards Apollonius.

This problem, originating before the beginning of the Christian era is remarkable in the history of Greek algebra since it is a problem in indeterminate analysis of the second degree, a problem the solution of which is more complicated than that of any in extant works of Diophantus.

K. B. Mollweide (who was born in Wolfenbüttel) is quoted as authority for the statement that his friend Gauss had solved the Cattle Problem completely.[2] If Gauss really obtained the solution he is the only person known to have done so. For, although it has been held very unsafe to claim that Archimedes had at his disposal no method sufficiently powerful to cope with such a question, no one contends (nor was it necessary for giving to Apollonius his quietus in accordance with Hultsch's suggestion) that he actually carried through all the necessary computations.

As final reference to the literature of our topic I note three papers published in America, two by A. H. Bell,[3] and one by Mansfield Merriman[4] to which the student who is unable to read the admirable and comprehensive German memoirs of Krumbiegel and Amthor will naturally turn. Bell's papers set forth the results of nearly four years of computation by himself and two others who constituted the "Hillsboro Mathematical Club" of Hillsboro, Ill. They computed thirty or thirty-one of the left-hand figures and twelve of the right-hand figures for each of the eight unknowns, as well as for the total number of the Cattle of the Sun. Amthor seems to have contented himself with computing the first four of the left-hand figures of (*a*) one of the unknowns, (*b*) the total (although similar values for other unknowns follow readily from his work). The corresponding printed results of Bell and Amthor do not agree.

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[1] *Quæstiones Archimedeæ*, l. c., and *Philologus*, Göttingen, Band 43, 1884, p. 486.

With these authorities Heath too seems to range himself. It is interesting to compare his discussion of the Cattle Problem in the first edition of his *Diophantos of Alexandria*, Cambridge, 1885, pp. 7, 142–147, with that given in the second.

[2] Hermann, *l. c.*, p. 230 of *Opuscula*.

[3] (1) "On the celebrated 'Cattle Problem' of Archimedes," *The Mathematical Magazine*, Washington, January, 1895, Vol. 2, pp. 163–164; (2) "The 'Cattle Problem' by Archimedes 251 B. C.," AMERICAN MATHEMATICAL MONTHLY, May, 1895, Vol. 2, pp. 140–141.

[4] M. Merriman, "The Cattle Problem of Archimedes," *Popular Science Monthly*, 1905, Vol. 67, pp. 660–665.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

COMMUNICATIONS FOR THIS DEPARTMENT SHOULD BE SENT TO H. BLUMBERG, University of Illinois.

SOME DRAWINGS AND GRAPHICAL SOLUTIONS IN NAVIGATION.

By WM. H. ROEVER.

1. *Introductory Note.*—During the past summer the U. S. Naval Auxiliary Reserve School, Municipal Pier, Chicago, made arrangements whereby its students were to take a course in navigation at the University of Chicago. As a member of the teaching staff at the University during the summer, I offered my services as instructor in navigation to these students. Professor Wilczynski and I were asked to give the work in nautical astronomy. Most of the students, while very intelligent, had only a limited knowledge of mathematics,—not extending beyond plane trigonometry,—and no knowledge whatever of astronomy. For this reason I at first made frequent appeal to their geometric intuition and solved by graphical processes the spherical triangles which arise in the determination of time, latitude, longitude, azimuth, and in the problem of great-circle sailing. I also drew a number of pictures, illustrating space relations. These pictures and graphical processes proved to be so helpful to my students that it seemed worth while to those who had charge of the course in Navigation at the University of Chicago to have them put in a form whereby they will be available to others. Hence this article. It should not be assumed from what has been said, that I did not also give the customary methods of computation.

2. *Time Determination.*—The determination of time reduces itself to the mathematical problem of finding an angle (the hour angle) of a spherical triangle of which the three sides (complements of the latitude, altitude and declination) are known. To solve graphically such a triangle let us consider the trihedral formed by the planes of the sides of the spherical triangle. Fig. 1 represents such

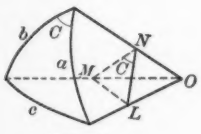

FIG. 1.

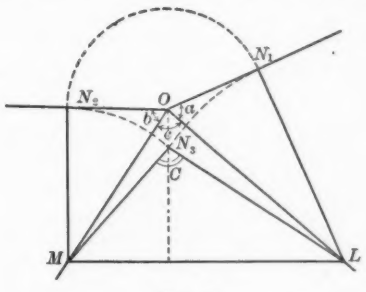

FIG. 2.

a trihedral in which a, b, c denote the sides of the spherical triangle or the face angles of the trihedral, and C denotes the required angle of the spherical triangle

or the corresponding dihedral angle of the trihedral. The point O represents the center of the sphere or the vertex of the trihedral, and N represents any point on the edge of the required dihedral angle. At N perpendiculars are erected to the edge ON in the faces which meet in ON. These perpendiculars meet the other edges of the trihedral in the points L and M. Then, in the triangle represented by LMN, the angle at N is equal to the required angle C of the spherical triangle. To find C graphically, let us cut the trihedral along the edge ON and open it up, or develop it, as shown in Fig. 2. Then N will assume the two positions N_1 and N_2, which lie at equal distances from O on the two positions assumed by the edge ON. The perpendiculars erected to these positions at the points N_1 and N_2, respectively, meet the new positions of the other edges in L and M, respectively. Hence ML, LN_1 and MN_2 in Fig. 2 are the true lengths of the sides of the triangle

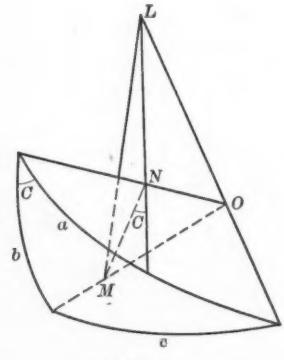

FIG. 3.

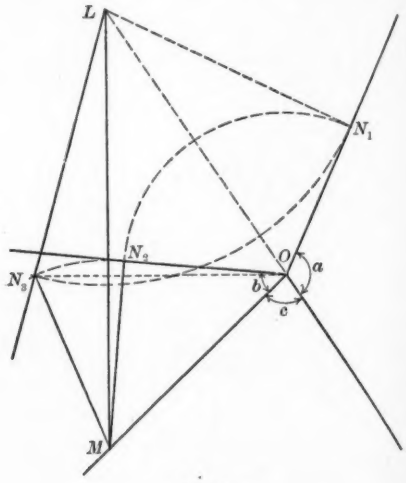

FIG. 4.

represented by LMN in Fig. 1. Thus we are led to the *Construction:* With O (Fig. 2) as vertex lay off the angle c equal to the side of the spherical triangle which lies opposite to the required angle C. Adjacent to the angle c and on either side of it lay off the angles a and b equal to the other two given sides of the spherical triangle. Then on the terminal lines of these last two angles take points N_1 and N_2 at equal distances from O. At these points erect perpendiculars N_1L and N_2M to these sides, meeting the sides of the angle c in the points L and M, respectively. With L as center and LN_1 as radius draw a circle, and with M as center and MN_2 as radius draw another circle. These two circles intersect in a point N_3, which, together with the points L and M forms the triangle represented by LMN in Fig. 1, and hence the angle LN_3M of Fig. 2 is the required angle C of the given spherical triangle. A check on the construction is furnished by the fact that the points N_3 and O must lie on the same perpendicular to the line LM.

In navigation, the sides b and c (complements of the latitude and altitude, respectively) are always acute, but the side a (complement of the declination) may be acute or obtuse. For the case in which the side a is obtuse the point L will lie on the produced edge, as shown in Fig. 3, and then the required angle C is the supplement of the angle MNL of the triangle MLN. The corresponding construction is shown in Fig. 4.

If the sides a and b of the given spherical triangle are nearly 90°, the points L and M of the constructions shown in Figs. 2 and 4, will fall beyond the limits of the drawings. In this case the construction explained in § 4 can be used to advantage.

3. *Solution of the triangle in the Saint Hilaire Method.*—To find the latitude and longitude of a ship by the Sumner Method, two *lines of position* must be determined. The Saint Hilaire Method for finding such lines, involves a spherical triangle of which two sides and the included angle are known and of which the side opposite the given angle is required.[1] To solve graphically this spherical triangle, let us consider Fig. 5, in which the edge of the given dihedral angle C

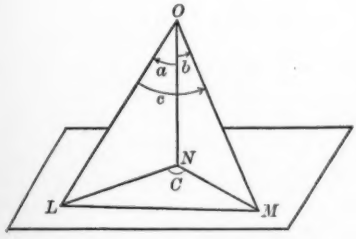

FIG. 5.

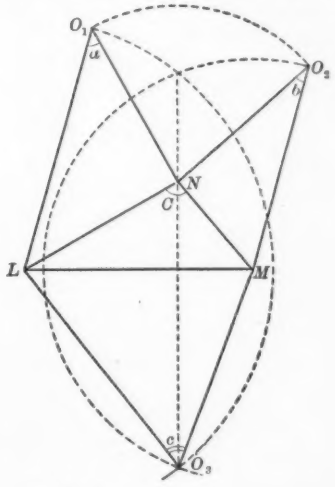

FIG. 6.

of the corresponding trihedral is represented as being in a vertical position, *i. e.*, perpendicular to the plane (let us say) of a table top. The points in which the edges of the trihedral meet this plane are represented by the points N, L, and M, of which the first lies on the edge of the given dihedral angle C. The center of the sphere, or the vertex of the trihedral, is represented by the point O. Let us now think of this trihedral as cut along its three edges ON, OM, OL and

[1] This side is the computed zenith distance of the observed object. The azimuth is usually obtained from an azimuth table, but it can easily be found graphically as shown in the *Remark* at the end of § 4.

of its faces as turned down around the lines NM, ML, LN into the plane of the table top. They then assume the position shown in Fig. 6. Hence the *Construction:* With N (Fig. 6) as vertex lay off the angle C equal to the given angle of the spherical triangle. At N erect perpendiculars to the sides of this angle and on these take points O_1 and O_2 at equal distances from N. Through O_1 and O_2 draw lines making with O_1N and O_2N the angles a and b, respectively, equal to the given sides of the spherical triangle, and meeting the sides of the angle C in the points L and M, respectively. Then with L as center and LO_1 as radius draw a circle, and with M as center and MO_2 as radius draw another circle. These circles meet in a point O_3, which with L and M forms the triangle represented in

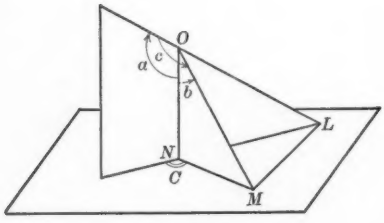

FIG. 7.

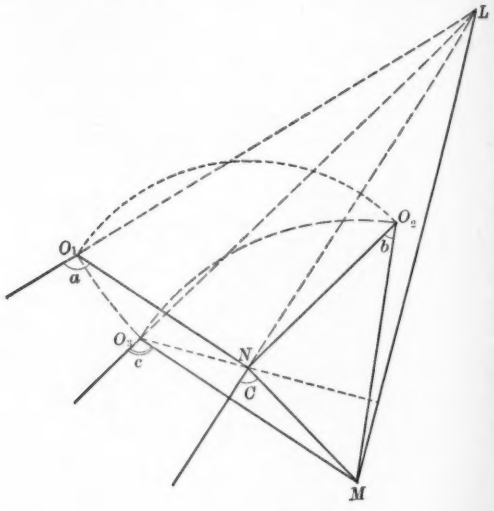

FIG. 8.

Fig. 5 by OLM and of which the angle at O is equal to the required side c of the spherical triangle. Hence the angle LO_3M of Fig. 6 is the required angle c of the spherical triangle. A check is furnished by the fact that O_3 and N should lie on the same perpendicular to LM.

In the navigation problem the side b (complement of the latitude) is always acute, but the parts C and a may be acute or obtuse. In the case where the part a is obtuse the point L lies on the produced edge, as shown in Fig. 7, and then the required part c is the supplement of the angle LOM in the triangle LOM. The corresponding construction is shown in Fig. 8.

If the parts a and b are nearly 90° the points L and M of Figs. 6 and 8 will fall beyond the limits of the drawings, and in this case the construction of § 4 can be used.

4. *Constructions available in failing cases of the above.*—Constructions which have the advantage over the constructions just given, in that all the points re-

main within the limits of the drawings, will now be given. These constructions are perfectly general and may be used whenever the given parts of the spherical triangle are less then 180°. These constructions will now be given, but for their proof the reader is referred to *Loria, Vorlesungen ueber Darstellende Geometrie*, Vol. II, p. 6.

Construction (Fig. 9 for a acute, Fig. 10 for a obtuse).—The three parts a, b, c (*i. e.*, the sides of the spherical triangle or the face angles of the corresponding trihedral) are laid off from a point O as shown in Figs. 9 and 10. A circle Γ of center O and any convenient radius is then drawn cutting the sides of the angles a, b, c in the points $\bar{C}$, $\bar{A}$, $\bar{B}_1$ and $\bar{B}_2$ as shown in the figures. From the points $\bar{B}_1$ and $\bar{B}_2$ perpendiculars are dropped to the rays $O\bar{C}$ and $O\bar{A}$, respectively,

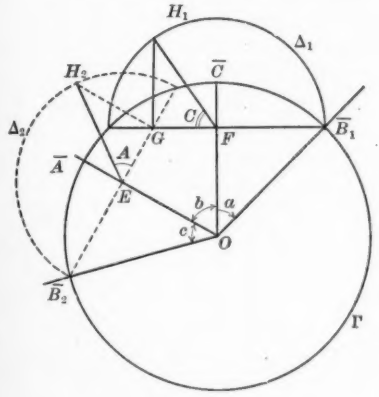

FIG. 9

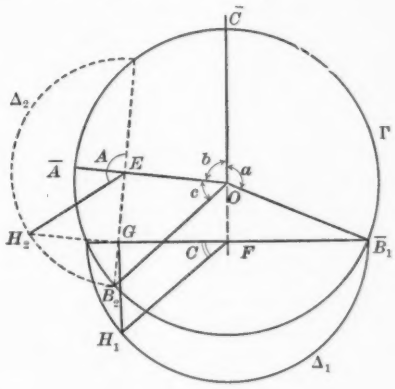

FIG. 10.

cutting these in the points F and E, respectively, and each other in the point G. With F as center and $F\bar{B}_1$ as radius a circle Δ_1 is drawn. At G a perpendicular is erected to the line $\bar{B}_1F$ cutting the circle Δ_1 in H_1. Then the supplement of the angle $\bar{B}_1FH_1$ is the required angle C of the spherical triangle of which the sides are the angles a, b, c.

Construction (Fig. 9 for a acute, Fig. 10 for a obtuse).—If a, b, C are the given parts of the spherical triangle, the circle Γ, the perpendicular $\bar{B}_1F$ and the circle Δ_1 are drawn as in the above preceding construction. The point H_1 is now taken on the circle Δ_1, so that the angle $\bar{B}_1FH_1$ is the supplement of the given angle C. From H_1 a perpendicular is dropped to $F\bar{B}_1$, meeting it in G. From G a perpendicular is dropped to $O\bar{A}$ meeting it in E, and meeting the circle Γ in $\bar{B}_2$. The angle $\bar{A}O\bar{B}_2$ is then equal to the required side c of this spherical triangle.

Remark.—In order to find the angle A (which is the azimuth in the nautical problem), we have merely to draw the circle Δ_2 of center E and radius $E\bar{B}_2$. At G we then erect a perpendicular to $\bar{B}_2E$ cutting Δ_2 in H_2. The supplement of the angle $\bar{B}_2EH_2$ is then equal to the required angle A.

5. *The great-circle-sailing problem.*—The problem of great-circle sailing consists in determining the latitude and longitude of various points Q of the great circle which connects two given points A and B on the surface of the earth. The points Q are then plotted on a Mercator chart[1] from which the course of the ship between consecutive points is determined. To devise a graphical solution of this problem let us first think of the points A and B as both situated in the same hemisphere (Fig. 11).

Now think of a plane tangent to the earth at its pole O' and upon this plane let us project the points A and B from the center O of the earth, denoting the projections by A' and B', respectively. Then the straight line $A'B'$ of this plane is the projection of the great circle path AB, and the lines $O'A'$ and $O'B'$ are the

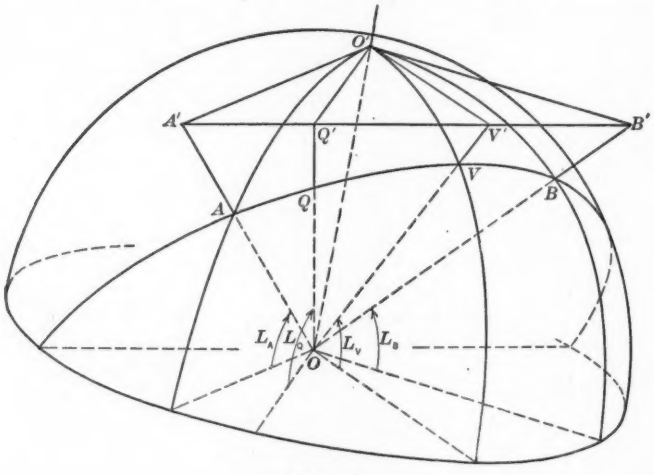

FIG. 11.

projections of the meridians of A and B. Let V denote the point of the great circle AB which is nearest the pole O'. Evidently the projection of the point V is the foot V' of the perpendicular from O' to $A'B'$ in the triangle $O'A'B'$. A general point Q of the great circle AB projects into a point Q' of the line $A'B'$. The tetrahedron $OA'B'O'$ represented in Fig. 11 is the inverted position of the tetrahedron $OLMN$ represented in Fig. 5, so that the parts a, b, C of Fig. 5 correspond respectively to the co-latitude ($90°$-L_A) of A, the co-latitude ($90°$-L_B) of B and the difference in longitude (equal to the angle $B'O'A'$) of the points A and B. Hence the part c of Fig. 5 corresponds to the angular distance AOB of Fig. 11. Thus the construction of Fig. 6 determines graphically (in that it determines the part c) the angular distance between the two points A and B when the latitudes and the difference in longitudes of these points are known. In order to find the latitude, longitude and distance from A (or from B) of the

[1] For the definition of Mercator projection see § 7.

point V and of the general point Q, we will begin by repeating the construction of Fig. 6. In Fig. 12 this much is shown by the heavily drawn lines; the points O', A', B', O_A, O_B, O''' of Fig. 12 correspond to the points N, L, M, O_1, O_2, O_3, respectively, of Fig. 6. In Fig. 12 the triangle $O'A'B'$ is the true shape of the triangle which is shown in Fig. 11 by the same letters. Hence in Fig. 12 the foot of the perpendicular from O' to $A'B'$ is the point V', and any point of $A'B'$ is a point Q'. The angles $A'O'''V'$ and $A'O'''Q'$ are then the angular distances of the points V and Q, respectively, from the point A. Evidently the angle $V'O'A'$

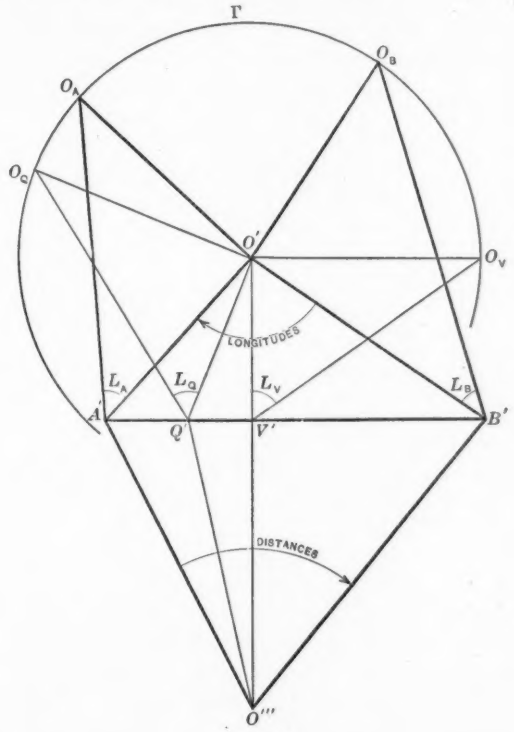

FIG. 12.

(Fig. 12) is the difference in longitude between the points V and A, and the angle $V'O'Q'$ is the difference in longitude between V and Q. To find the latitudes of V and Q let us first draw a circle Γ of center O' and radius $O'O_A = O'O_B$ (equal to the distance represented by OO' in Fig. 11). To get the latitude of V draw in Fig. 12 a line through O' perpendicular to $O'V'$ and cutting Γ in O_v. Then the angle $O'V'O_v$ is the latitude of V. Similarly, to get the latitude of Q draw a line through O' perpendicular to $O'Q'$ and cutting Γ in O_Q. The angle $O'Q'O_Q$ is then the latitude of Q. This is evident from the fact that the triangles $O'O_vV'$ and

$O'O_QQ'$ of Fig. 12 are the revolved positions (turned into the plane of the triangle $O'A'B'$) of the triangles represented by $O'V'O$ and $O'Q'O$ in Fig. 11.

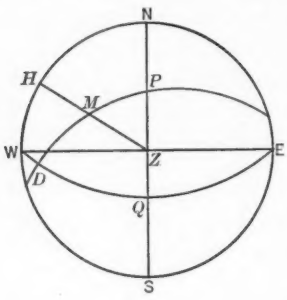

FIG. 13.

Just as we started with Fig. 6 the construction of Fig. 12 corresponding to the case where A and B are in the same hemisphere, so for the case where A and $B$are in different hemispheres we start the construction with Fig. 8.

6. *Definition of a good picture.*—In the absence of models, good pictures serve very well to convey an adequate notion of space relations. By a good picture of a space object is meant a plane representative which, when properly placed, produces upon the retinal surface of the eye an image which does not differ much from that produced by the object itself in a position in which one is accustomed to see or think of the object. It can easily be shown that a central

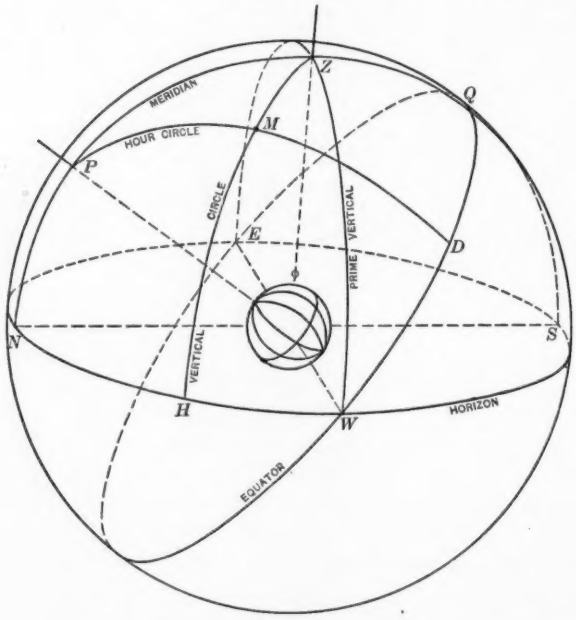

FIG. 14.

or parallel (orthographic or oblique) projection in general satisfies this criterion[1] and therefore such a projection of an object in an accustomed position furnishes a

[1] See for instance, pages 148 and 152 of the article on Descriptive Geometry in Vol. 25 of this MONTHLY.

good picture. What is here meant by an accustomed position will be made clear from the statements which follow. A stereographic projection of a sphere is a central projection for a particular position of the center of projection.[1] However, the stereographic projection of the celestial sphere with its circles of reference does

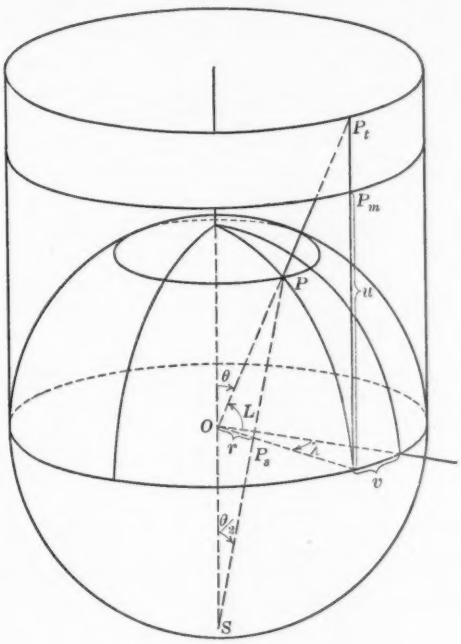

FIG. 15.

not produce nearly so good a picture as does an orthographic projection on a general plane. This will be evident to the reader when a comparison is made of Figs. 13 and 14 which are stereographic and orthographic projections, respectively. Hence a stereographic projection, although it is a projection which satisfies the criterion of producing the same retinal image as the object itself, does not furnish a good picture, because it is projected from a point from which one is not accustomed to seeing the sphere.

7. *Stereographic and Mercator projections defined.*—We have already used the terms Mercator projection and stereographic projection. For the purpose of defining these, as well as showing a simple relation which exists between them, we will make use of the foregoing picture (Fig. 15) which is an orthographic projection.

The stereographic projection of a sphere is the central projection of the points of the surface of the sphere upon a diametral plane from a pole of the great circle

[1] For the definition of stereographic projection see § 7.

in which this plane cuts the sphere. In Fig. 15 the plane of projection is the equatorial plane, and the center of projection is the south pole. The projection of the point P of the sphere is the point P_s of the equatorial plane.

The Mercator projection of a sphere is the development of a right circular cylinder, tangent to the sphere along the equator, the points of which are obtained from those of the sphere in the following manner. The point P of the sphere is projected from the center O into a point P_t of the cylinder (Fig. 15). Upon the element of the cylinder which passes through the point P_t the Mercator projection P_m of the point P lies, and its distance u from the equator is equal to the Naperian logarithm of the reciprocal $(1/r)$ of the distance (r) of the stereographic projection P_s from the center O of the sphere, the radius of which is taken to be unity.

Thus if of a point P of the sphere, L = latitude and λ = longitude, and θ is the co-latitude, so that

$$\theta = \frac{\pi}{2} - L \text{ and hence } \theta/2 = \pi/4 - L/2,$$

and if for the stereographic projection P_s the polar coördinates are r and λ, and for the Mercator projection the rectangular coördinates are u and v, then

$$\begin{cases} r = \tan\frac{\theta}{2} = \tan\left(\frac{\pi}{4} - \frac{L}{2}\right), \\ \lambda = \lambda, \end{cases}$$

and

$$\begin{cases} u = \log_e \frac{1}{r} = -\log_e \tan\frac{\theta}{2} = -\log_e \tan\left(\frac{\pi}{4} - \frac{L}{2}\right), \\ v = \lambda. \end{cases}$$

Thus one easily sees the simple relation which exists between these two types of map.

8. *Mean sun and equation of time defined.*—In the definitions of mean time and the equation of time, a good picture showing the relation which exists between the mean and apparent suns is very helpful. In the orthographic projection shown in Fig. 16 the celestial sphere with the equinoctial and ecliptic are represented. In this figure, the vernal equinox is represented by the point r, the position which the true sun occupies at perihelion (which occurs about January first) is represented by S_0, and the position which the true sun occupies at any particular instant (the position shown in the figure corresponds to about February 12) is represented by S. Before defining the mean sun, let us first define a fictitious sun which moves in the ecliptic at a uniform rate in the same direction as the true sun does and coincides with the true sun at perihelion (and therefore also at aphelion). The point F represents the position which this sun occupies when the true sun is at S. The *mean* sun may now be defined as the point which moves along the equinoctial at a uniform rate and coincides with the fictitious sun at the vernal equinox (and therefore also at the autumnal equinox).

The position which the mean sun occupies when the true sun is at S is represented by M. In other words, the right ascension of the mean sun M is equal to the mean celestial longitude of the fictitious sun F, these two artificial suns being in coincidence at the vernal equinox. Fig. 16 also represents the center O and the north pole P of the celestial sphere and the hour circles of S, F and M. The hour circles of S and F intersect the equinoctial in the points

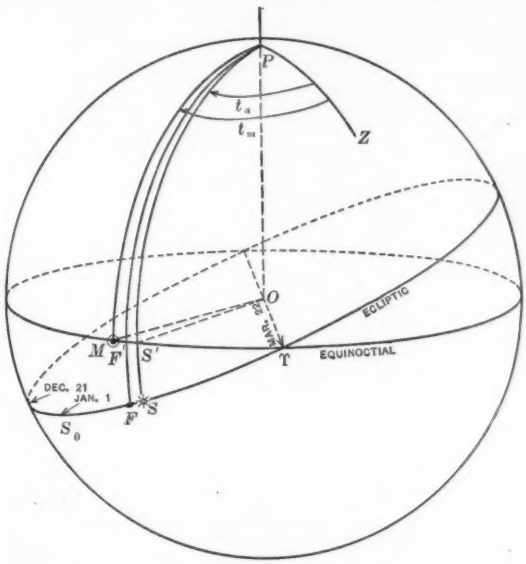

FIG. 16.

S' and F', respectively. The difference in right ascension between S and M, *i. e.*, the angle MOS' is called the equation of time. It is the difference between mean and apparent solar times. Thus if Z represents the zenith of a particular place of the earth at a particular instant,

t_m, = angle ZPM, is the local mean time of this place,
t_a, = angle ZPS, is the local apparent solar time of this place,

and

$$\epsilon, = t_m - t_a = \text{angle } MOS', \text{ is the equation of time.}^1$$

9. *The Sumner Method.*—In describing the Sumner Method of determining the position of a ship at sea, a picture representing both the celestial and the terrestrial spheres proves to be very helpful. In Fig. 17 such a picture is represented in orthographic projection. The positions on the celestial sphere, of the pole, zenith and two positions of the sun are represented by P, Z, S_1, S_2, respectively. The positions on the terrestrial sphere, of the points immediately below

[1] It should not be forgotten that hour angle (time) and right ascension are measured in opposite directions.

these, are denoted by p, z, s_1, s_2, respectively. Then z is the position of the ship. and s_1, s_2 are called the subsolar (or in the case of a star, the substellar) points, The meridian of Greenwich is shown on both of the spheres as is also the equator. It is thus evident that the terrestrial longitude of a subsolar point is equal to the Greenwich apparent time and that the latitude is equal to the declination of the true sun. A small circle of the terrestrial sphere which passes through the position of the ship (z) and has as pole the subsolar point is called a Sumner circle. It is

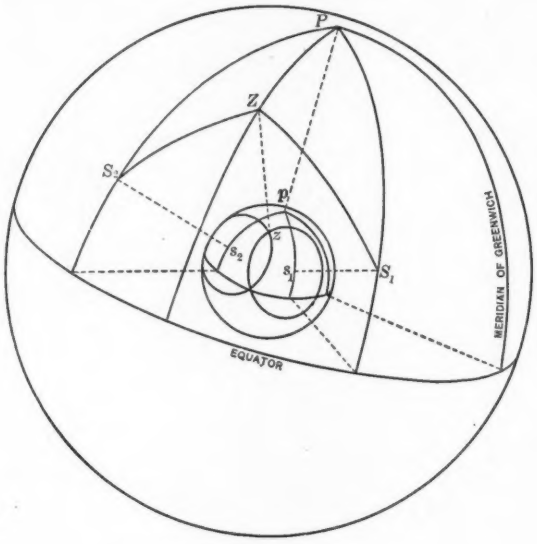

FIG. 17.

easy to see from the figure that the angular radius of this circle is equal to the zenith distance of the sun. The figure shows the two Sumner circles which correspond to S_1 and S_2 and which intersect in the position z of the ship.

10. *The Gnomonic Chart.*—By a gnomonic chart is meant the projection of the points of a spherical surface upon a tangent plane to this surface from the center of the sphere. A straight line of such a chart corresponds to a great circle of the sphere. Such charts are used in solving graphically the problem of great-circle sailing, of which one graphical solution has already been given in § 5. In fact the projection $O'A'B'$ (Fig. 11) on the plane tangent to the sphere at the point O' is a gnomonic chart of the triangle $O'AB$ of the sphere for the case where the plane of projection is tangent to the sphere at the pole O'. In this case the meridians project into straight lines radiating from the point of tangency and the parallels of latitude project into concentric circles whose common center is the point of tangency. If, however, the plane of projection is tangent to the sphere at a point not coincident with a pole, the meridians project into straight lines which

radiate from the point where the axis of the earth pierces the plane of projection, and the parallels of latitude project into conics. This is the case because the planes of the meridians all pass through the axis, which contains the center of projection, and because the projecting rays of a parallel of latitude form a right circular cone. Fig. 18 enables one to see this more clearly. If on such a chart a sufficient number of meridians and parallels of latitude are represented, it is an easy matter to read off from the chart the latitude and longitude of various points of the straight line representative of a great-circle path of the sphere. The method of doing this as well as that of finding the distance and course (direction of sailing) is explained on the great-circle charts which are published by the U. S. Hydrographic Office.

The construction of a gnomonic chart is identical with that of a horizontal sun-dial for a place whose latitude is equal to that of the point of tangency of the plane of projection of this chart. This can be easily seen by the aid of Fig. 18, in which we may now consider the horizontal table top as the dial plate and the axis of the sphere as the gnomon, or style, of the dial. The lines representing the meridians are then positions of the shadow of the style upon the dial plate, and the conics representing the parallels of latitude are the diurnal paths of the

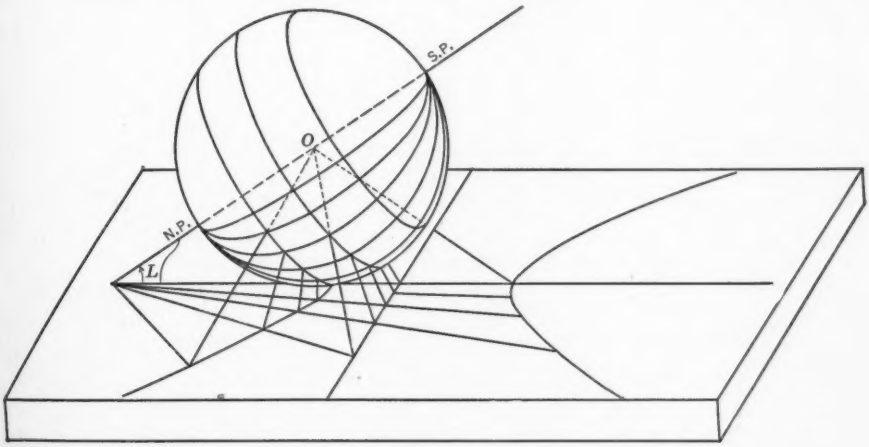

FIG. 18.

shadow of a knob (situated at the point O) of the style on the dial plate. In the dialing problem it is required to find the position of the shadow of the style for the hours of the day. The diurnal paths of the knob are the conics lying between the conics representing the Tropics of Capricorn and Cancer. It is on these conics that the equation of time is laid off from the position of the shadow of the style at apparent solar noon in order to make possible the use of a sundial for the determination of mean time.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

To the list of graduate students who have been granted the Doctorate in mathematics at American Universities since June, 1917, as printed in the September number of this MONTHLY, should be added the following: H. E. BRAY, Rice Institute, "A Green's theorem in terms of Lesbesgue's integral"; TERESA COHEN, Johns Hopkins, "An investigation of plane quartic curves"; ANNA M. HOWE, Cornell, "The classification of plane involutions of order three"; L. J. ROUSE, Michigan, "A contribution to the question of linear dependence in linear integral equations."

After the present issue of this journal was in type the EDITOR received from Professor E. R. HEDRICK, Chairman of the Association's Dictionary Committee, a few remarks concerning the article of Professor MILLER on pages 383 to 387 of this issue (seen by Professor HEDRICK in galley proof). These remarks are as follows: "While the committee on a Mathematical Dictionary has taken no steps concerning Professor Miller's article and while the material was not submitted to it for approval, the chairman of the committee feels sure that the committee would unquestionably sanction a statement approving the style and the nature of the contents of Professor Miller's article. The only possible question that suggests itself is concerning the length to be assigned to each department in order to keep the dictionary as a whole within reasonable limits. No final decision has been arrived at concerning this matter. Sample statements of this character should be of greatest assistance to the committee and it is suggested that others follow Professor Miller's example in submitting definitions."

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VOLUME XXV DECEMBER, 1918 NUMBER 10

THE CONTENT OF A SECOND COURSE IN CALCULUS.[1]

By E. J. MOULTON, Northwestern University.

The second course in calculus, as presented in different colleges and by different instructors, varies considerably in content. In view of its position on the one hand as often the culmination of the study of mathematical analysis both for students specializing in science and engineering and for students who will be teachers of mathematics in our secondary schools, and on the other hand as a necessary prerequisite for much of the later work in mathematics, it merits careful consideration. A general discussion of the course at a meeting like this should prove stimulating to all of us who have been faced with its difficulties; and it is my desire to learn the views of other teachers as much as to argue my own views that has led me to speak on this subject to-day.

Before considering the content of the course we should think of what its general aim should be. This general aim in turn depends upon the students who are expected to take the course, their preparation and their requirements. As to preparation, the minimum may be assumed to consist of courses in trigonometry, college algebra, analytical geometry and a three-hour year course in calculus. This is doubtless quite generally exceeded, but closely approximates the most common actual prerequisites for the second calculus course; and it is not desirable to assume a much better preparation. In my own classes, for instance, many of the students have had, or are taking, a course in mechanics, but I find it impossible to assume that course as a part of the student's preparation. At best the student's outlook on the field of mathematics is very restricted, and even his working knowledge of calculus is limited to the very simplest portions.

[1] This paper is a revision of one presented to the Mathematical Association of America at the meeting in Chicago on December 27, 1917.